

# Absence of Arbitrage and Martingale Pricing

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# Outline

Background and Fundamental Theorem of Asset Pricing

SPD and EMM

Dividends and Intermediate Consumption

Redundant Securities

Complete Markets

The Black-Scholes Model

Arbitrage-Free Term Structure Models

# Roadmap

- ▶ Today: Bridge between formal background and applications.
- ▶ Building blocks of discrete-time AP, reintroduced in continuous time.
  - ▶ SDF (referred to here as SPD), risk-neutral measure (EMM), no arbitrage, pricing equation, complete markets.
  - ▶ But not just redefinitions: some concepts (and complications) new to continuous time.
- ▶ Where we're headed when we return:
  - ▶ Consumption and portfolio choice under the martingale approach.
  - ▶ Equilibrium & CAPMs (CCAPM, ICAPM).

# Background: FTAP in Discrete vs. Continuous Time

## Fundamental Theorem of Asset Pricing (Discrete Time):

No arbitrage  $\Leftrightarrow$  Existence of state prices, state price density, equivalent martingale measure

## Main differences from discrete time:

1. Existence of a SPD or an EMM (formally defined shortly) does not preclude arbitrage.
  - ▶ Arbitrages can be constructed using doubling strategies, so we'll have to restrict trading.
2. Absence of arbitrage does not imply existence of a SPD or an EMM.
  - ▶ Risk premia need to satisfy some technical conditions for SPD & EMM to exist.

But can show SPD & EMM existence (and properties) are equivalent:

## Proposition (Continuous Time)

SPD  $\Leftrightarrow$  EMM

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# Setting: The Securities Market Model

Assume:

1. a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,
2. a time interval  $\mathcal{T} = [0, T]$ ,
3. a Brownian motion  $Z = (Z_1, \dots, Z_d)$  on  $(\Omega, \mathcal{F}, \mathbb{P})$ ,
4. the standard filtration  $\mathbb{F}$  of  $Z$ ,
5.  $N + 1$  securities indexed by  $n = 0, \dots, N$ .
  - (a) Security 0 is the “riskless” security (bank account). Its price at time 0 is  $B_0 \equiv S_{0,0} = 1$ , and its price at time  $t$  is determined by  $dB_t = r_t B_t dt$ , where  $r$  is the short rate process.
  - (b) Securities  $1, \dots, N$  are the “risky” securities. Their prices follow an Itô process:

$$d \begin{pmatrix} S_{1,t} \\ \dots \\ S_{N,t} \end{pmatrix} = \mu_t dt + \sigma_t dZ_t,$$

where  $\mu \in (\mathcal{L}^1)^N$  and  $\sigma \in (\mathcal{L}^2)^{N \times d}$ .

Denote  $S = (S_0, \dots, S_N)$  and  $S_{1N} = (S_1, \dots, S_N)$ . For now, assuming no dividends and no intermediate consumption. (Considering some investor, not necessarily rep. agent.)

# Trading Strategies

## Definition

A **trading strategy** is a process  $\theta \in \mathcal{L}(S)$ , where  $\mathcal{L}(S) = \{\theta \in \mathcal{L}^{\mathcal{N}} : \theta\mu \in \mathcal{L}^1, \theta\sigma \in (\mathcal{L}^2)^d\}$ .

- ▶  $\theta_{1N} = (\theta_1, \dots, \theta_N)$ : vector of risky-asset holdings (itself a process).
- ▶ Define gains from trading:

$$\int_0^t \theta_s dS_s = \int_0^t (\theta_{0,s} r_s B_s + \theta_{1N,s} \mu_s) ds + \int_0^t \theta_{1N,s} \sigma_s dZ_s.$$

## Definition

A trading strategy is **self-financing** iff

$$\theta_t S_t = \theta_0 S_0 + \int_0^t \theta_s dS_s.$$

- ▶ Represents dynamic budget constraint (with no dividends or intermediate consumption).
- ▶ Discrete-time equivalent:  $(\theta_t - \theta_{t-1})S_t = 0 \Rightarrow \theta_t S_t = \theta_0 S_0 + \sum_{s=0}^{t-1} \theta_s (S_{s+1} - S_s)$ .

# Cash Flows

## Definition

A **cash flow** is a pair  $(C_0, C_T)$  where  $C_0 \in \mathbb{R}$  and  $C_T$  is  $\mathcal{F}_T$ -measurable.

## Definition

A self-financing trading strategy  $\theta$  **finances** a cash flow  $(C_0, C_T)$  iff  $C_0 = -\theta_0 S_0$  and  $C_T = \theta_T S_T$ .

Note  $C$  here is cash flow generated by trading securities (can think of as consumption, but will later add separate endowment process).

## Definition

A cash flow is **marketable** iff it is financed by some trading strategy  $\theta$ .

Denote by  $M$  the set of marketable cash flows.



# Arbitrage, SPD, and EMM

## Definition

An **arbitrage** is a marketable cash flow such that  $C_0 \geq 0$ ,  $C_T \geq 0$ , and either  $C_0 > 0$  or  $\mathbb{P}(C_T > 0) > 0$ .

## Definition

A **state price density (SPD)** is a strictly positive Itô process  $\pi$  with  $\pi_0 = 1$ , such that  $\pi S$  is a martingale,  $\pi_t S_t = E_t[\pi_T S_T]$ .

## Definition

An **equivalent martingale measure (EMM)** is a probability measure  $\mathbb{Q}$  such that  $\hat{S} \equiv S/B$  is a martingale under  $\mathbb{Q}$ , and  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$ .

Think of  $\pi$  as the SDF and  $\mathbb{Q}$  as the risk-neutral measure. (Makes clear why existence is equivalent.)

EMM:  $\hat{S}_t = E_t^Q[\hat{S}_T] \Leftrightarrow S_t/B_t = E_t^Q[S_T/B_T] \Leftrightarrow S_t = E_t^Q\left[\exp\left(-\int_t^T r_s ds\right) S_T\right]$ . Will see this repeatedly.

# Doubling Strategies

Doubling strategy generates arbitrage opportunities in continuous time, because:

- ▶ Trading can take place an arbitrarily large number of times over finite horizon.
- ▶ Wealth can become arbitrarily negative (no bankruptcy restrictions yet).

Possible restrictions to rule out doubling strategies:

1. Trading can take place only a finite number of times  $\rightarrow$  trading strategies have to be **simple**.  
(Recall  $\int_0^t \theta_s d\hat{S}_s$  is a martingale if  $\theta_t$  is simple.) Seems undesirable.

2. Restrict  $\theta \in \mathcal{H}^2(\hat{S})$ :

$$E \left[ \int_0^T (\theta_t \hat{\mu}_t)^2 dt \right] < \infty \quad \text{and} \quad E \left[ \int_0^T (\theta_t \hat{\sigma}_{i,t})^2 dt \right] < \infty, \quad i = 1, \dots, d.$$

(Rules out very fat-tailed trading strategies, and turns stochastic integral into a martingale.)

3. Impose debt constraint so that discounted wealth has to stay above some threshold:

$$\underline{\Theta}(\hat{S}) \equiv \{\theta : \exists k \text{ s.t. } \theta_t \hat{S}_t \geq k \forall t\}.$$

(Recall that a local martingale that is bounded below is a supermartingale.)

Note that bounded short rate  $\Rightarrow \underline{\Theta}(\hat{S}) = \underline{\Theta}(S)$ ,  $\mathcal{H}^2(\hat{S}) = \mathcal{H}^2(S)$ .

# From SPD/EMM to No Arbitrage

## Theorem

Suppose that an EMM  $\mathbb{Q}$  exists.

- ▶ If trading strategies belong to  $\mathcal{H}^2(\hat{S})$ , and the Radon-Nikodym derivative  $d\mathbb{Q}/d\mathbb{P} \in L^2$ , then there is no arbitrage.
- ▶ If trading strategies belong to  $\underline{\Theta}(\hat{S})$ , then there is no arbitrage.
- ▶ First part: Discounted prices are martingales  $\Rightarrow$  discounted gains are martingales  $\Rightarrow -\hat{C}_0 = E^{\mathbb{Q}}[\hat{C}_T]$ .
- ▶ Second part: Local martingale being bounded below turns it into a supermartingale, so  $\hat{C}_0 + E^{\mathbb{Q}}[\hat{C}_T] \leq 0$ .

# From SPD/EMM to No Arbitrage

Combining previous result with the fact that  $\text{SPD} \Leftrightarrow \text{EMM}$ :

## Corollary

Suppose that an SPD  $\pi$  exists.

- ▶ If trading strategies belong to  $\mathcal{H}^2(\hat{S})$ , and  $\pi_T B_T \in L^2$ , then there is no arbitrage.
- ▶ If trading strategies belong to  $\underline{\Theta}(\hat{S})$ , then there is no arbitrage.

# From Security Prices to EMM

We now know how to go from EMM to no arbitrage. How do we know whether an EMM exists?

Need conditions on price processes so that risk premia “make sense” (and meet technical conditions):

## Theorem (EMM Existence Conditions and Characterization)

If the following hold jointly:

1.  $\hat{\mu}_t - \hat{\sigma}_t \eta_t = 0$  has a solution  $\eta_t$ ;
2.  $\hat{\sigma}_t \in (\mathcal{H}^2)^{N \times d}$ ;
3.  $\eta$  satisfies Novikov's condition;

then an EMM  $\mathbb{Q} = Q^\eta$  exists, with  $dQ^\eta / DP = \xi_T^\eta$  and  $\xi_t^\eta = \exp(-\int_0^t \eta'_s dZ_s - \frac{1}{2} \int_0^t \|\eta_s\|^2 ds)$ . The SPD also exists, and is given by  $\pi_t = \xi_t^\eta / B_t$ .

- ▶ Last slide of Lecture 1 (from Girsanov's Theorem):  $v_t = \mu_t - \sigma_t \eta_t$  is drift of  $S$  under  $Q^\eta \Rightarrow$  set to 0 for discounted process  $\hat{S}$  to get (discounted) martingale pricing.
- ▶ EMM to SPD:  $\hat{S}_t = E_t^Q[\hat{S}_T] = \frac{E_t[\xi_T^\eta \hat{S}_T]}{\xi_t^\eta} \implies \pi_t S_t = E_t[\pi_T S_T]$ .

# From Security Prices to EMM

Economic interpretation: the  $n$ 'th “component” of  $\hat{\mu}_t - \hat{\sigma}_t \eta_t = 0$  is

$$\frac{\mu_{n,t}}{S_{n,t}} - r_t = \sum_{i=1}^d \frac{\sigma_{n,i,t}}{S_{n,t}} \eta_{i,t}.$$

- ▶ Equivalently: risk premium for security  $n = \frac{\sigma_{n,t}}{S_{n,t}} \eta_t = \left( \frac{dS_{n,t}}{S_{n,t}} \right) \left( -\frac{d\xi_t^\eta}{\xi_t^\eta} \right)$ , or  $-\text{Cov}_t(\text{return}, \text{SDF})$ .
- ▶ Thus the risk premium of security  $n$  is the sum over Brownian motions of the security's loading on each Brownian motion, times the Brownian motion's risk premium ( $\eta_{i,t}$ ).
- ▶ This is also a necessary condition for an EMM to exist and for absence of arbitrage:

## Proposition

Suppose that equation  $\hat{\mu}_t - \hat{\sigma}_t \eta_t = 0$  does not have a solution. Then there are arbitrages, both for trading strategies in  $\mathcal{H}^2(\hat{S})$  and in  $\underline{\Theta}(\hat{S})$ .

If there were no solution, then there would exist two (portfolios of) securities that would have the same loadings on the Brownian motions but different risk premia, leading to an arbitrage.

# From Security Prices to EMM

- ▶ Interpretation for the EMM existence conditions:
  1. The risk premia of the securities should be “consistent,” in that they can be derived from risk premia of the underlying Brownian motions. (Restatement of first half of previous slide.)
  2. Security prices cannot be too volatile.
  3. The risk premia of the Brownian motions must satisfy Novikov’s condition (cannot be too large).
- ▶ Interpretation of the EMM:
  - ▶ From Itô’s Lemma,  $d\xi_t^\eta = -\xi_t^\eta \eta'_t dZ_t$ .
  - ▶  $\xi_t^\eta$  is the ratio of the probabilities under  $\mathbb{Q}$  and  $\mathbb{P}$  of a given Brownian motion path up to time  $t$ .
  - ▶  $\mathbb{Q}$  will put more weight than  $\mathbb{P}$  on high marginal utility states, where  $\xi^\eta$  rises.
- ▶ Interpretation of the SPD  $\pi_t = \xi_t^\eta / B_t$ :
  - ▶ Again from Itô’s Lemma,  $d\pi_t = -\pi_t r_t dt - \pi_t \eta'_t dZ_t$ .
  - ▶ Drift term: SPD decreases over time (incorporates discounting). Diffusion term: same interpretation as for EMM.

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# Adding Dividends and Intermediate Consumption

- ▶ Now extend the securities market model to incorporate dividends and intermediate consumption.
- ▶  $\delta = (0, \delta_1, \dots, \delta_N)$  is the rate at which the securities pay dividends;  $\delta \in (\mathcal{L}^1)^{N+1}$ .
- ▶  $D$  is the cumulative dividend process:

$$D_t = \int_0^t \delta_s ds.$$

- ▶  $G$  is the gain process:

$$G = S + D.$$

- ▶  $c$  is the rate at which consumption occurs;  $c \in \mathcal{L}^1$ .
- ▶ The analysis from before generalizes, including intermediate cash flows and replacing the price process by the gain process. Will skip through quickly.

# Definitions with Intermediate Cash Flows $\leftrightarrow$

## Definition

A **trading strategy** is a process  $\theta \in \mathcal{L}(G)$ .

## Definition

A **cash flow** is a triplet  $(C_0, c, C_T)$  where  $C_0 \in \mathbb{R}$ ,  $c \in \mathcal{L}^1$ , and  $C_T$  is  $\mathcal{F}_T$ -measurable.

## Definition

A trading strategy  $\theta$  **finances** a cash flow  $(C_0, c, C_T)$  iff

$$\theta_t S_t = \theta_0 S_0 + \int_0^t \theta_s dG_s - \int_0^t c_s ds,$$

$C_0 = -\theta_0 S_0$ , and  $C_T = \theta_T S_T$ .

## Definition

An **arbitrage** is a marketable cash flow such that  $C_0 \geq 0$ ,  $c \geq 0$ ,  $C_T \geq 0$ , and  $C_0 > 0$  or  $\mathbb{P}(c > 0) > 0$  or  $\mathbb{P}(C_T > 0) > 0$ .

## SPD and EMM with Intermediate Cash Flows $\leadsto$

The **SPD-adjusted** gain process  $G^\pi$ :

$$G_t^\pi = \pi_t S_t + \int_0^t \pi_s \delta_s ds$$

The **discounted** gain process  $\widehat{G}$ :

$$\widehat{G}_t = \frac{S_t}{B_t} + \int_0^t \frac{\delta_s}{B_s} ds$$

### Definition

A **SPD** is a strictly positive Itô process  $\pi$  such that  $G^\pi$  is a martingale and  $\pi_0 = 1$ .

### Definition

An **EMM** is a probability measure  $\mathbb{Q}$  such that  $\widehat{G}$  is a martingale under  $\mathbb{Q}$  and  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$ .

## Pricing with Intermediate Cash Flows $\leadsto$

### Theorem

Suppose that an EMM  $\mathbb{Q}$  exists. If trading strategies belong to  $\mathcal{H}^2(\hat{G})$ , and the Radon-Nikodym derivative  $d\mathbb{Q}/d\mathbb{P} \in L^2$ , then there is no arbitrage. Alternatively, if trading strategies belong to  $\underline{\Theta}(\hat{G})$ , then there is no arbitrage.

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# Redundant Securities

We've obtained an analogue to the pricing equation for  $N$  primitive securities (e.g., for  $N = 1$ , think of risky security as a stock). Want to know whether & how to price other assets (e.g., derivatives).

Assume trading strategies  $\theta \in \mathcal{L}(S)$  are such that there is no arbitrage.

## Definition

A self-financing trading strategy  $\theta \in \mathcal{L}(S)$  **replicates** an  $\mathcal{F}_T$ -measurable random variable  $C_T$  iff  $C_T = \theta_T S_T$ .

## Definition

A new security with time  $T$  payoff  $\tilde{S}_T$  is **redundant** iff there exists an admissible trading strategy  $\tilde{\theta}$  that replicates  $\tilde{S}_T$ .

## Proposition

For a redundant security,  $\tilde{S}_t = \tilde{\theta}_t S_t$ .

# FTAP in Continuous Time

## Proposition (Risk-Neutral Pricing)

For a redundant security,

$$\tilde{S}_t = E_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) \tilde{S}_T \right].$$

Tells us how to price any portfolio of securities by no arbitrage.

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# Complete Markets

To apply arbitrage pricing, need to characterize the set of cash flows that can be replicated. Start by defining and characterizing the case where *all* cash flows can be replicated: complete markets.

## Definition

Markets are **complete** iff all  $\mathcal{F}_T$ -measurable random variables  $C_T$  such that  $C_T/B_T \in L^2(Q)$  can be replicated by a trading strategy in  $\mathcal{H}^2(\hat{S})$  or (if  $C_T \geq 0$ ) by a trading strategy in  $\underline{\Theta}(\hat{S})$ .

## Theorem

Markets are complete iff  $\text{rank}(\sigma_t) = d$  a.s.

- ▶ Similar to discrete time: there, completeness  $\Leftrightarrow$  linearly independent securities = # of nodes one period ahead; here, # of nodes replaced by # of Brownian motions.
- ▶ Can replicate infinite-dimensional set of cash flows with finite securities, since trading can happen infinitely often.

## Theorem

Markets are complete iff the EMM is unique.

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## Black-Scholes: Setup

Can now begin applying FTAP. First consider best-known application: option pricing. Model:

- ▶ Two securities: bond with  $B_0 = 1$  and constant short rate,

$$dB_t = rB_t dt,$$

and stock that follows geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$

where  $Z_t$  is one-dimensional standard Brownian motion.

- ▶ Itô's Lemma: Discounted price process is  $d\hat{S}_t = (\mu - r)\hat{S}_t dt + \sigma \hat{S}_t dZ_t$ .
- ▶ EMM existence conditions are met straightforwardly, with price of risk for the Brownian motion of  $\eta = \frac{\mu - r}{\sigma}$  and Radon-Nikodym derivative of EMM  $\mathbb{Q}$  w.r.t.  $\mathbb{P}$  of

$$\xi_T = \exp\left(-\eta Z_T - \frac{1}{2}\eta^2 T\right).$$

- ▶ Girsanov's Theorem: Brownian motion under  $\mathbb{Q}$  is  $Z_t^{\mathbb{Q}} = Z_t + \eta t$ , and  $dS_t = rS_t dt + \sigma S_t dZ_t^{\mathbb{Q}}$ .
- ▶ Given  $\sigma > 0$ , markets are complete, and the EMM is unique.

## Pricing: The Martingale Approach

- ▶ We want to price an option. Usual approach: replicate security's payoffs and calculate expectation under EMM.
- ▶ So consider a new security with a time  $T$  payoff  $\tilde{S}_T \in L^1(Q)$ , and set  $\tilde{S}_T = G(S_T)$ . Then

$$\tilde{S}_t = E_t^Q [\exp(-r(T-t)) G(S_T)].$$

- ▶ Since  $dS_t = rS_t dt + \sigma S_t dZ_t^Q$ , solving this GBM gives:

$$S_T = S_t \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) (T-t) + \sigma (Z_T^Q - Z_t^Q) \right].$$

- ▶ So denoting a standard normal RV by  $\tilde{\epsilon}$ ,

$$\tilde{S}_t = \exp(-r(T-t)) E \left[ G \left( S_t \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) (T-t) + \left( \sigma \sqrt{T-t} \right) \tilde{\epsilon} \right] \right) \right].$$

- ▶ Can now explicitly compute the expectation for **European call option**  $G(S_T) = \max(S_T - K, 0)$ .

## Pricing: The Martingale Approach

- Denote call price by  $C(S_t, t)$ . From pricing formula on previous slide:

$$C(S_t, t) = \exp(-r(T-t))E \left[ \max \left\{ S_t \exp \left[ \left( r - \frac{1}{2}\sigma^2 \right) (T-t) + \left( \sigma\sqrt{T-t} \right) \tilde{\epsilon} \right] - K, 0 \right\} \right].$$

- RV inside expectation is 0 when  $S_T = S_t \exp \left[ \left( r - \frac{1}{2}\sigma^2 \right) (T-t) + \left( \sigma\sqrt{T-t} \right) \tilde{\epsilon} \right] \leq K$ , i.e.:

$$z_2 \equiv \frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \leq -\tilde{\epsilon}.$$

- So  $C(S_t, t) = \exp(-r(T-t))E[S_T \mathbb{1}_{\{z_2 \geq -\tilde{\epsilon}\}}] - \exp(-r(T-t))KE[\mathbb{1}_{\{z_2 \geq -\tilde{\epsilon}\}}]$ , or:

$$C(S_t, t) = \exp(-r(T-t)) \underbrace{E[S_T \mathbb{1}_{\{z_2 \geq -\tilde{\epsilon}\}}]}_{C_1} - \exp(-r(T-t))KN(z_2),$$

where  $N(\cdot)$  is the standard normal CDF. First term  $C_1$  is the expectation of truncated lognormal, can solve by more or less brute force...

## Pricing: The Martingale Approach

- For  $S_T = S_t \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) (T - t) + \left( \sigma \sqrt{T - t} \right) \tilde{\epsilon} \right]$  and  $C_1 = E[S_T \mathbb{1}_{\{z_2 \geq -\tilde{\epsilon}\}}] = E[S_T \mathbb{1}_{\{\tilde{\epsilon} \geq -z_2\}}]$ , using standard normal PDF:

$$\begin{aligned} C_1 &= \frac{1}{\sqrt{2\pi}} \int_{-z_2}^{\infty} S_t \exp \left[ \left( r - \frac{1}{2} \sigma^2 \right) (T - t) + \left( \sigma \sqrt{T - t} \right) \tilde{\epsilon} \right] \exp \left( -\frac{1}{2} \tilde{\epsilon}^2 \right) d\tilde{\epsilon} \\ &= S_t \exp(r(T - t)) \frac{1}{\sqrt{2\pi}} \int_{-z_2}^{\infty} \exp \left[ -\frac{1}{2} \left( \tilde{\epsilon} - \sigma \sqrt{T - t} \right)^2 \right] d\tilde{\epsilon} \\ &= S_t \exp(r(T - t)) \frac{1}{\sqrt{2\pi}} \int_{-z_2 - \sigma \sqrt{T - t}}^{\infty} \exp \left( -\frac{1}{2} \tilde{\epsilon}^2 \right) d\tilde{\epsilon} \\ &= S_t \exp(r(T - t)) N(z_1), \end{aligned}$$

where  $z_1 \equiv z_2 + \sigma \sqrt{T - t}$ .

- Putting together:

$$C(S_t, t) = S_t N(z_1) - \exp(-r(T - t)) K N(z_2).$$

- Can solve for put price similarly, or using put-call parity (replicate payoff of put using bond, call, and stock).

# Black-Scholes Formula

## Proposition

For a European call  $C(S_t, t)$  and a European put  $P(S_t, t)$  with strike price  $K$  and maturity  $T$ , we have

$$C(S_t, t) = S_t N(z_1) - \exp(-r(T-t)) K N(z_2),$$

and

$$P(S_t, t) = \exp(-r(T-t)) K N(-z_2) - S_t N(-z_1).$$

where  $N(\cdot)$  is the CDF of the standard normal distribution,

$$z_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and  $z_2 = z_1 - \sigma\sqrt{T-t}$ .

Notice expected stock return  $\mu$  doesn't show up! We solved by no arbitrage using that option is redundant. (Expected stock return does show up implicitly in  $S_t$ .)

## Pricing: The PDE Approach

- ▶ Martingale approach for solving Black-Scholes — calculate expectation under EMM of discounted payoff — is, in my view, generally most straightforward.
- ▶ But the original approach to no-arbitrage pricing was to derive and solve a PDE for the redundant security. (Upside: Better interpretation of price in terms of replicating strategy.)
- ▶ Time  $T$  payoff of the redundant security: still  $G(S_T)$ . Time  $t$  price:  $g(S_t, t)$ , and assume  $g(S, t)$  is twice continuously differentiable.
- ▶ The discounted price  $e^{-rt}g(S_t, t)$  is a martingale under  $\mathbb{Q}$ :  $E_t^{\mathbb{Q}}[d(e^{-rt}g(S_t, t))] = 0$ .
- ▶ Applying Itô's Lemma, we get

$$-g(S_t, t)r + \mathcal{D}_S g(S_t, t) + g_t(S_t, t) = 0,$$

where  $\mathcal{D}_S g(S_t, t) = g_S(S_t, t)rS_t + \frac{1}{2}g_{SS}(S_t, t)\sigma^2 S_t^2$ .

- ▶ This, along with the terminal condition  $g(S, T) = G(S)$ , is the famous Black-Scholes PDE. Can be rewritten as  $g(S_t, t)r - g_S(S_t, t)rS_t = g_t(S_t, t) + \frac{1}{2}g_{SS}(S_t, t)\sigma^2 S_t^2$ . LHS is return from long-call, short-stock position; RHS is “theta” (time decay) plus “gamma” (convexity w.r.t. stock price).



# Replication Under the PDE Approach

- ▶ The pricing of redundant securities rests on their replicability, which can be made explicit by continuing logic of previous slide.
- ▶ For a security with time  $T$  payoff  $G(S_T)$  and time  $t$  price  $g(S_t, t)$ , the replicating strategy consists in holding  $(\theta_t^S, \theta_t^B)$ :

$$\theta_t^S \equiv g_S(S_t, t)$$

shares of the stock, and

$$\theta_t^B B_t \equiv g(S_t, t) - g_S(S_t, t) S_t$$

dollars in the bond.

- ▶ Can then show that the strategy  $(\theta^B, \theta^S)$  satisfies:
  1. self-financing;
  2. replicates  $G(S_T)$ ;
  3. in  $\mathcal{L}(B, S)$ ;
  4. the stochastic integral  $\int_0^t \theta_s^S d\widehat{S}_s$  is a martingale under  $\mathbb{Q}$ .

# Outline

Background and Fundamental Theorem of Asset Pricing

SPD and EMM

Dividends and Intermediate Consumption

Redundant Securities

Complete Markets

The Black-Scholes Model

**Arbitrage-Free Term Structure Models**

# Term-Structure Models: Overview

- ▶ Arbitrage-free term structure models tend to behave better in continuous time (closed-form solutions, negativity issues, ...), so are a good place to illustrate/apply techniques so far.
- ▶ Start with general one-factor model: Assume that under  $\mathbb{Q}$  the short rate process is

$$dr_t = \mu(r_t, t) dt + \sigma(r_t, t) dZ_t^Q.$$

- ▶  $P(t, T)$ : time- $t$  price of the zero-coupon bond maturing at time  $T$  with face value 1:

$$P(t, T) = E_t^Q \left[ \exp \left( - \int_t^T r_s ds \right) \right].$$

- ▶ Setting  $P(t, T) = f(r_t, t)$  for some  $f$ , can then derive and solve PDE.
- ▶ Next slide shows how to apply the above to the Vasicek model of the term structure (nice illustration of power of PDE approach in this case).
- ▶ You'll cover term-structure models in more detail next week with Kerry (and saw Vasicek model already in recitation with J.R.), so I'll stop with this high-level overview.
- ▶ When we come back: portfolio choice using the martingale approach, and then CAPMs.

## Example: Vasicek Model $\leadsto$

- ▶ Vasicek model setup (Ornstein–Uhlenbeck model for short rate under  $\mathbb{Q}$ ):

$$dr_t = -\theta(r_t - \bar{r}) dt + \sigma dZ_t^{\mathbb{Q}}.$$

- ▶ Using that discounted price is a martingale under  $\mathbb{Q}$  and Itô's Lemma, obtain bond price PDE:

$$rf = f_t - f_r \theta(r - \bar{r}) + f_{rr} \frac{1}{2} \sigma^2.$$

- ▶ Guess exponentially affine solution:

$$f(r, t) = \exp(a_0(t) + a_1(t)r).$$

- ▶ Get a system of ODEs for  $a_0(t)$  and  $a_1(t)$ :

$$\begin{aligned}\dot{a}_0 &= -\bar{r}a_1 - \frac{\sigma^2}{2}a_1^2 \\ \dot{a}_1 &= 1 + \theta a_1\end{aligned}$$

with boundary conditions  $a_0(T) = 0$ ,  $a_1(T) = 0$ .  $a_0(t)$  and  $a_1(t)$  can then be solved analytically.