

Portfolio Choice and Equilibrium in Continuous Time

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Outline

Background

Portfolio Choice: Martingale Approach

Equilibrium Setup

CAPMs in Continuous Time

Recap & Roadmap

What we've done:

1. Math.
2. Risk-neutral pricing in continuous time.
 - ▶ Expected excess returns depend on “risk premium” on Brownian motions η_t : $\frac{\mu_{n,t}}{S_{n,t}} - r_t = \frac{\sigma_{n,t}}{S_{n,t}} \eta_t$.
 - ▶ Haven't yet said where these fundamental risk premia come from: have taken them to be exogenous, and priced derivatives on (or portfolios of) assets given exogenous premia.

What we'll do now:

- ▶ Consumption and portfolio choice for an individual trader (still with exogenous risk premia).
- ▶ Then define and explore equilibrium in continuous time with many traders.
- ▶ Can then connect economic fundamentals **endogenously** to risk premia and portfolio choice.
- ▶ Will also give us a better understanding of pricing under \mathbb{P} in continuous time.

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Two approaches to portfolio choice:

1. **Martingale approach** (today).

- ▶ Use martingale pricing equation to solve for optimal **payoffs**.
- ▶ Most powerful in complete markets: if all payoffs are attainable, can solve a **static** choice problem.
- ▶ But will discuss how it extends to incomplete markets (briefly today, in detail in a few lectures).
- ▶ Downside: have to work backwards from optimal payoffs to figure out optimal trading strategy.

2. Dynamic programming approach (next lecture).

- ▶ More “traditional” approach: solve instantaneous choice problem recursively.
- ▶ Often more tractable in incomplete markets (esp. for partial equilibrium choice problems).
- ▶ Provides direct characterization of optimal trading strategy.
- ▶ Downside: produces nonlinear PDE for value function, which can generally be solved in closed form only in special cases.

Reminder: The Securities Market Model

For now, same setting as before:

1. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
2. a time interval $\mathcal{T} = [0, T]$,
3. a Brownian motion $Z = (Z_1, \dots, Z_d)$ on $(\Omega, \mathcal{F}, \mathbb{P})$,
4. the standard filtration \mathbb{F} of Z ,
5. $N + 1$ securities indexed by $n = 0, \dots, N$ (denote $S = (S_0, \dots, S_N)$ and $S_{1N} = (S_1, \dots, S_N)$).
 - (a) Security 0 is the “riskless” security (bank account). Its price at time 0 is $B_0 \equiv S_{0,0} = 1$, and its price at time t is determined by $dB_t = r_t B_t dt$, where r is the short rate process.
 - (b) Securities $1, \dots, N$ are the “risky” securities. Their prices follow an Itô process:

$$d \begin{pmatrix} S_{1,t} \\ \dots \\ S_{N,t} \end{pmatrix} = \mu_t dt + \sigma_t dZ_t,$$

where $\mu \in (\mathcal{L}^1)^N$ and $\sigma \in (\mathcal{L}^2)^{N \times d}$.

Continuing to assume no dividends for now, but will add these when considering equilibrium.

Will allow for intermediate consumption throughout.

Portfolio Choice: Setup

- ▶ We'll now add an investor's objective function, and consider that investor's portfolio choice and consumption problem.

Definition

A **consumption plan** is a pair (c, C_T) where $c \in \mathcal{L}$, C_T is \mathcal{F}_T -measurable, $c \geq 0$, and $C_T \geq 0$.

- ▶ Assume the investor has wealth W in period 0 and time-additive expected utility:

$$U(c, C_T) = E \left[\int_0^T u_t(c_t) dt + U_T(C_T) \right],$$

where the functions u_t and U_T are strictly increasing and concave.

- ▶ Assume directly that trading strategies are such that $\int_0^t \theta_s d(S_s/B_s)$ is a martingale under Q .

Portfolio Choice: Setup

Definition

A consumption plan is **feasible** iff the cash flow $(-W, c, C_T)$ is marketable. Denote by \mathcal{C} the set of feasible consumption plans.

- Recall C_0 defined previously s.t. $C_0 = -\theta_0 S_0$, so above implies $W = \theta_0 S_0$.

The Investor's Problem, \mathcal{P}

$$\begin{aligned} \max_{c, C_T} U(c, C_T), \\ (c, C_T) \in \mathcal{C} \end{aligned}$$

Definition

A consumption plan (c, C_T) is **optimal** iff it solves \mathcal{P} . A trading strategy is **optimal** iff it finances $(-W, c, C_T)$, where (c, C_T) is an optimal consumption plan.

Martingale Approach to Portfolio Choice: The Static Problem

- ▶ \mathcal{P} is stated as a choice problem over **consumption** (or **payoff**) outcomes.
- ▶ Can be solved either using martingale approach or dynamic programming (in which case the problem, and choice variables, need to be reformulated).
- ▶ With complete markets, \mathcal{P} is equivalent to a simple **static** problem as follows, which shows why the martingale approach tends to be more powerful in this case.

Proposition

When markets are complete, the problem \mathcal{P} is equivalent to the problem \mathcal{P}_Q :

$$\begin{aligned} \max_{c, C_T} U(c, C_T), \quad \text{subject to} \\ E^Q \left[\int_0^T \frac{c_t}{B_t} dt + \frac{C_T}{B_T} \right] = W, \\ c \geq 0, \quad C_T \geq 0. \end{aligned}$$

- ▶ Infinite horizon: Same setup, but $T = \infty$. Complication: \mathbb{Q} is no longer equivalent to \mathbb{P} on \mathcal{F}_∞ ; Huang and Pagès (1992) give conditions to make things work (lower bound on wealth & enough discounting).

Martingale Approach: Optimal Consumption

- ▶ Can now solve the static problem \mathcal{P}_Q to get optimal consumption (then portfolio choice).
- ▶ First: Restate budget constraint in \mathbb{P} rather than \mathbb{Q} :

$$W = E^Q \left[\int_0^T \frac{c_t}{B_t} dt + \frac{C_T}{B_T} \right] = E \left[\int_0^T \frac{\xi_t}{B_t} c_t dt + \frac{\xi_T}{B_T} C_T \right] = E \left[\int_0^T \pi_t c_t dt + \pi_T C_T \right].$$

- ▶ Denote by $i_t(y)$ the inverse of u'_t and by $I_T(y)$ the inverse of U'_T . Then Lagrangian FOC is

$$c_t^* = i_t(\lambda \pi_t), \quad C_T^* = I_T(\lambda \pi_T). \quad (\dagger)$$

- ▶ To determine the Lagrange multiplier λ , we can use the static budget constraint under \mathbb{P} :

$$E \left[\int_0^T \pi_t i_t(\lambda \pi_t) dt + \pi_T I_T(\lambda \pi_T) \right] = W. \quad (\star)$$

Proposition

Suppose there exists a λ solving equation (\star) . Then the solution to \mathcal{P}_Q is given by (\dagger) .

Optimal Trading Strategy

- ▶ This tells us optimal **consumption outcomes**, but we've skipped past how to implement these via dynamic trading.
- ▶ Given consumption solution, now construct the trading strategy that finances (W, c^*, C_T^*) .
- ▶ Consider the wealth W_t^* required to finance the optimal consumption plan from time t on:

$$W_t^* = B_t E_t^Q \left[\int_t^T \frac{c_s^*}{B_s} ds + \frac{C_T^*}{B_T} \right] = \frac{1}{\pi_t} E_t \left[\int_t^T \pi_s i_s(\lambda \pi_s) ds + \pi_T I_T(\lambda \pi_T) \right].$$

- ▶ Assume that $W_t^* = F(\pi_t, S_t, t)$ (so that the relevant state variables are (π_t, S_t)). PDE for F :

$$c_t^* - r_t F(\pi, S, t) + \mathcal{D}_{\pi S} F(\pi, S, t) + F_t(\pi, S, t) = 0,$$

$$\mathcal{D}_{\pi S} F = F_\pi \pi_t (-r_t + \eta_t' \eta_t) + F_{S'} r_t S_t + \frac{1}{2} (\pi_t^2 \eta_t' \eta_t F_{\pi\pi} - 2\pi_t F_{\pi S'} \sigma_t \eta_t + \text{tr}(\sigma_t \sigma_t' F_{SS'})),$$

with terminal condition $F(\pi, S, T) = C_T^*$.

- ▶ Why? $\frac{F(\pi_t, S_t, t)}{B_t} + \int_0^t \frac{c_s^*}{B_s} ds$ is Q -martingale, since it's $E_t^Q(X)$ with $X = \int_0^T \frac{c_t^*}{B_t} dt + \frac{C_T^*}{B_T}$. Use this with Itô + Girsanov + SPD evolution to get above PDE. Not great to work with!

Optimal Trading Strategy

- ▶ Martingale representation:

$$E_t^Q(X) = E^Q(X) + \int_0^t \frac{-F_\pi(\pi_s, S_s, s)\pi_s\eta'_s + F_{S'}(\pi_s, S_s, s)\sigma_s}{B_s} dZ_s^Q.$$

- ▶ Accordingly, risky asset investment is defined by

$$(\theta_{1N,t}^*)\sigma_t = -F_\pi(\pi_t, S_t, t)\pi_t\eta'_t + F_{S'}(\pi_t, S_t, t)\sigma_t,$$

which has a unique solution iff $N = d$:

$$[\theta_{1N,t}^*]' = -F_\pi(\pi_t, S_t, t)\pi_t(\sigma_t\sigma_t')^{-1}(\mu_t - r_t S_t) + F_S(\pi_t, S_t, t).$$

- ▶ Riskless asset investment is the remainder:

$$F(\pi_t, S_t, t) = \theta_{0,t}^* B_t + \theta_{1N,t}^* S_t.$$

- ▶ This is unwieldy, so am mostly skipping over in class. Will return to optimal trading strategy in more detail using dynamic programming.

Optimal Payoffs: A Special Case

To get some closed-form solutions for consumption, now assume:

1. a constant investment opportunity set: $r_t = r$, $\mu_t = I_{S_t} \bar{\mu}$, and $\sigma_t = I_{S_t} \bar{\sigma}$, where I_{S_t} is a $N \times N$ diagonal matrix with $I_{S_t}(n, n) = S_{n,t}$.
 - implies $\frac{dS_{n,t}}{S_{n,t}} = \bar{\mu}_n dt + \sum_{i=1}^d \bar{\sigma}_{n,i} dZ_{i,t} \Rightarrow$ i.i.d. returns.
2. CRRA utility: $u_t(c_t) = bu(c_t)e^{-\beta t}$ and $U_T(C_T) = u(C_T)e^{-\beta T}$, with $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

► Solution:

$$c_t^* = \left(\frac{\lambda \pi_t}{be^{-\beta t}} \right)^{-\frac{1}{\gamma}}, \quad C_T^* = \left(\frac{\lambda \pi_T}{e^{-\beta T}} \right)^{-\frac{1}{\gamma}}.$$

► λ is just a scaling constant, but can be solved for explicitly using

$$E \left[\int_0^T \pi_t \left(\frac{\lambda \pi_t}{be^{-\beta t}} \right)^{-\frac{1}{\gamma}} dt + \pi_T \left(\frac{\lambda \pi_T}{e^{-\beta T}} \right)^{-\frac{1}{\gamma}} \right] = W.$$

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Setup

We can now consider equilibrium. To do so, we'll add endowments and dividends. Setup:

- ▶ Start as usual with a probability space (Ω, \mathcal{F}, P) , a time interval $\mathcal{T} = [0, T]$, a Brownian motion $Z = (Z_1, \dots, Z_d)$ on (Ω, \mathcal{F}, P) , and the standard filtration \mathbb{F} of Z .
- ▶ N securities with dividends paid at rate $\delta = (\delta_{1,t}, \dots, \delta_{N,t}) \in (\mathcal{L}^1)^N$ and terminal price $S_T = (S_{1,T}, \dots, S_{N,T})$. The (constant) supply of the securities is $x = (x_1, \dots, x_N)$.
- ▶ I agents, with preferences:

$$U_i(c_i, C_{i,T}) = E \left[\int_0^T u_{i,t}(c_{i,t}) dt + U_{i,T}(C_{i,T}) \right],$$

with $u_{i,t}$ and $U_{i,T}$ strictly increasing and concave. (Could also assume continuum of agents.)

- ▶ Endowment: Agent i receives an endowment of the consumption good at a rate $e_i \in \mathcal{L}^1$. Also has an endowment $\bar{\theta}_{i,0}$ of the securities at time 0; $\sum_{i=1}^I \bar{\theta}_{i,0} = x$.
- ▶ Useful to redefine notation slightly (without loss) for security price processes:

$$dS_t = I_{S_t} \bar{\mu}_t dt + I_{S_t} \bar{\sigma}_t dZ_t,$$

where $\bar{\mu} \in (\mathcal{L}^1)^N$, $\bar{\sigma} \in (\mathcal{L}^2)^{N \times d}$, and $I_{S_t} = \text{diag}(S_{1,t}, \dots, S_{N,t})$. Agents can also borrow and lend at riskless rate r_t (zero net supply). Again assume trading strategies in $\mathcal{L}(S)$ are s.t. $\int_0^t \theta_s d(S_s/B_s)$ is a Q -martingale.

Setup

Individual Agent's Problem, \mathcal{P}_i

$$\max_{c_i, C_{i,T}} U(c_i, C_{i,T}),$$

$$(c_i, C_{i,T}) \in \mathcal{C}_i,$$

where \mathcal{C}_i is the set of feasible consumption plans for agent i . A consumption plan $(c_i, C_{i,T})$ is optimal iff it solves \mathcal{P}_i . A trading strategy θ_i is optimal iff it finances the cash flow $(-\bar{\theta}_{i,0}S_0, c_i - e_i, C_{i,T})$.

Definition (Securities Market Equilibrium)

A **securities market (SM) equilibrium** is a price process S , a vector of trading strategies $(\theta_1, \dots, \theta_I)$, and consumption policies (c_1, \dots, c_I) , such that

1. optimality: (c_i, θ_i) is optimal for agent i ;
2. market clearing:

$$\sum_{i=1}^I \theta_i = x, \quad x\delta + \sum_{i=1}^I (e_i - c_i) = 0.$$

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Characterizing Equilibrium

- ▶ Assuming equilibrium existence, what can we say about the features of equilibrium?
- ▶ For simplicity, assume again that risky assets do not pay dividends.
- ▶ Equilibrium short rate process r : determined endogenously such that the instantaneous demand for riskless borrowing and lending is 0.
- ▶ No arbitrage in equilibrium \Rightarrow

$$\bar{\mu}_t - r_t 1 = \bar{\sigma}_t \eta_t$$

has a solution η_t . This solution is unique only when markets are complete.

- ▶ In non-vector form, as in last lecture, n 'th “component” of this is $\bar{\mu}_{n,t} - r_t = \sum_{j=1}^d \bar{\sigma}_{n,j,t} \eta_{j,t}$.
- ▶ Want to know how risk premia are determined in GE, without reference to exogenous η_t .
- ▶ We'll derive CAPM-type equations linking risk premia to covariances with aggregate variables.
 - ▶ We'll see that the continuous-time setting allows for very sharp and general statements.

Risk Premium

Before turning to CAPMs, review features of equilibrium risk premia.

- ▶ Given η_t , recall SPD evolves according to $d\pi_t = -\pi_t r_t dt - \pi_t \eta'_t dZ_t$.
- ▶ Using this and equation on previous slide, expression for risk premium follows (as on problem set):

$$\frac{E_t(dS_{n,t})}{S_{n,t}} - r_t dt = -\text{Cov}_t\left(\frac{dS_{n,t}}{S_{n,t}}, \frac{d\pi_t}{\pi_t}\right)$$

- ▶ As in last lecture: A security is valuable, and thus has a low risk premium, if it has a high payoff in high SPD states, i.e. in states where consumption is valuable.
- ▶ Analogue: $E_t(R_{n,t+1}) - R_{f,t+1} = -\text{Cov}_t\left(R_{n,t+1}, \frac{M_{t+1}}{E_t(M_{t+1})}\right)$.
- ▶ When markets are incomplete, EMM & SPD no longer unique. The above expression holds in this case for **any** admissible SPD.

Implication: Hansen-Jagannathan Bound

Hansen-Jagannathan Bound

An asset's instantaneous Sharpe ratio is bounded above by the volatility of the state-price density:

$$\left| \frac{\bar{\mu}_{n,t} - r_t}{\|\bar{\sigma}_{n,t}\|} \right| \leq \|\eta_t\|.$$

- ▶ As in discrete time: for a model to generate high Sharpe ratios (like we see in the data), need to have sufficiently high volatility of the SPD (or SDF).

CCAPM

- ▶ The above used a general SPD, but we can further characterize π_t in equilibrium. Our first “economic” CAPM will be the consumption CAPM (Breedon (1979)).
- ▶ When markets are complete, optimal consumption for agent i satisfies

$$u'_{i,t}(c_{i,t}^*) = \lambda_i \pi_t, \quad (\star)$$

and π_t is unique. Consumption growth is perfectly instantaneously correlated across agents, and marginal utility growth is identical across agents.

- ▶ When markets are incomplete, (\star) holds for some π , but perfect correlation across agents no longer guaranteed.
 - ▶ A different individual budget constraint is associated with each admissible SPD.
 - ▶ But one can show a duality result: problem \mathcal{P}_i is equivalent to the dual problem of (i) solving the complete-markets problem \mathcal{P}_i for each admissible SPD and (ii) minimizing the maximized utility across all SPDs. But this may imply a different SPD should be used in (\star) for each agent!
 - ▶ We'll discuss this duality result in detail in a few lectures.

- ▶ Let's apply our earlier risk premium result, $\frac{E_t(dS_{n,t})}{S_{n,t}} - r_t dt = -\text{Cov}_t\left(\frac{dS_{n,t}}{S_{n,t}}, \frac{d\pi_t}{\pi_t}\right)$.
- ▶ $u'_{i,t}(c_{i,t}^*) = \lambda_i \pi_t$ implies (Itô's Lemma) that $d\pi_t = \frac{1}{\lambda_i} d[u'_{i,t}(c_{i,t}^*)] = \frac{1}{\lambda_i} [u''_{i,t}(c_{i,t}^*) dc_{i,t}^* + \text{terms in } dt]$.
- ▶ Thus $\frac{E_t(dS_{n,t})}{S_{n,t}} - r_t dt = \left[-\frac{u''_{i,t}(c_{i,t}^*)}{u'_{i,t}(c_{i,t}^*)} \right] \text{Cov}_t\left(\frac{dS_{n,t}}{S_{n,t}}, dc_{i,t}^*\right)$. Divide through by term in brackets, sum across i , to obtain expression with respect to aggregate consumption c_t :

Consumption CAPM (CCAPM)

$$\frac{E_t(dS_{n,t})}{S_{n,t}} - r_t dt = A_t \text{Cov}_t\left(\frac{dS_{n,t}}{S_{n,t}}, dc_t\right),$$

where A_t is (proportional to) the harmonic mean of absolute risk aversion,

$$A_t = -\frac{1}{\sum_{i=1}^I \frac{u'_{i,t}(c_{i,t})}{u''_{i,t}(c_{i,t})}} > 0.$$

CCAPM

- ▶ Two advantages in continuous time relative to discrete time:
 1. The CCAPM holds even when markets are incomplete. (Thus uninsurable risk can't affect risk premia in any such diffusion model!)
 2. The CCAPM involves the covariance with consumption and not with the marginal utility of consumption. (Heuristically, over a small time interval, utility is approximately quadratic, so mean-variance analysis applies.)
- ▶ In a representative-agent economy (or a complete-market economy in which a representative agent can be constructed), the H-J bound implies that

$$\left| \frac{\bar{\mu}_{n,t} - r_t}{\bar{\sigma}_{n,t}} \right| \leq \gamma_t \sigma_{c,t},$$

where $\gamma_t = -c_t u_t''(c_t) / u_t'(c_t)$ (rep. agent RRA) and $\sigma_{c,t}$ is instantaneous volatility of consumption growth.

Generalizations: Non-Traded Assets, Heterogeneous Information

- ▶ The CCAPM holds for an asset only if all agents can trade in this asset without frictions.
- ▶ Heterogeneous information: One can sometimes obtain a CCAPM even if agents have heterogeneous information (Grossman and Shiller (1982)).
 - ▶ Euler equations hold with conditional expectation taken with respect to each agent's information set.
 - ▶ Thus (by law of iterated expectations), must also hold conditional on information set common to all agents.
 - ▶ For same CCAPM as above to hold, need A_t to belong to common info set. Can get this, e.g., if all agents have CRRA utility with the same risk aversion.

Intertemporal CAPM (ICAPM)

- ▶ Taking our calculations a different direction will allow us to derive our next CAPM, the ICAPM (Merton (1973)).
- ▶ Consider state variables X , and assume drift and diffusion of S_t depend only on X_t, t . Denote value function for agent i at time t : $V_i(W_{i,t}, X_t, t)$.
- ▶ Envelope theorem: $u'_{i,t}(c_{i,t}) = V_{i,W}(W_{i,t}, X_t, t)$.
- ▶ Thus $d\pi_t = \frac{1}{\lambda_i} d[V_{i,W}(W_{i,t}, X_t, t)] = \frac{1}{\lambda_i} [V_{i,WW}dW_{i,t} + (V_{i,WX})'dX_t + \text{terms in } dt]$, so that

$$\frac{E_t(dS_{n,t})}{S_{n,t}} - r_t dt = \left(-\frac{V_{i,WW}}{V_{i,W}} \right) \text{Cov}_t \left(\frac{dS_{n,t}}{S_{n,t}}, dW_{i,t} \right) + \left(-\frac{(V_{i,WX})'}{V_{i,W}} \right) \text{Cov}_t \left(\frac{dS_{n,t}}{S_{n,t}}, dX_t \right).$$

- ▶ Same steps as before lead to:

ICAPM

$$\frac{E_t(dS_{n,t})}{S_{n,t}} - r_t dt = A_t^W \text{Cov}_t \left(\frac{dS_{n,t}}{S_{n,t}}, dW_t \right) + A_t^X \text{Cov}_t \left(\frac{dS_{n,t}}{S_{n,t}}, dX_t \right).$$

ICAPM

- ▶ Risk premia depend on covariance of return with (i) aggregate wealth, and (ii) state variables, since the security can be used to hedge changes in the state variables.
- ▶ Advantages in continuous time:
 1. The ICAPM holds even when markets are incomplete.
 2. The ICAPM involves the covariance with wealth and not with the marginal utility of wealth.
 3. Effects of covariance with wealth and with state variables can be neatly separated in closed form.
- ▶ Special case of ICAPM: investment opportunity set is constant (constant X). Then ICAPM becomes the standard CAPM:

$$\begin{aligned}\frac{E_t(dS_{n,t})}{S_{n,t}} - r_t dt &= A_t^W \text{Cov}_t \left(\frac{dS_{n,t}}{S_{n,t}}, dW_t \right) \\ &= \underbrace{\frac{\text{Cov}_t \left(\frac{dS_{n,t}}{S_{n,t}}, \frac{dW_t}{W_t} \right)}{\text{Var}_t \left(\frac{dW_t}{W_t} \right)}}_{\beta_{n,t}} \times \underbrace{\gamma_t^W \text{Var}_t \left(\frac{dW_t}{W_t} \right)}_{\frac{E_t(dW_t)}{W_t} - r_t dt}.\end{aligned}$$

Example I: The Breeden-Lucas Economy

- ▶ Now specialize further: Pure exchange economy, parametric assumptions.
- ▶ Continuum of identical investors with isoelastic utility:

$$u_t(c) = e^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma}.$$

- ▶ A single risky asset with unit supply, dividend process:

$$d\delta_t = \theta_t \delta_t dt + \sigma_t \delta_t dZ_t.$$

- ▶ Risk-free bond with zero net supply.
- ▶ Implies state price density $u'_t(c_t) = \lambda \pi_t$. Since $c_t = \delta_t$, must have $\pi_t = e^{-\rho t} (\delta_t / \delta_0)^{-\gamma}$, and thus $\frac{d\pi_t}{\pi_t} = -r_t dt - \gamma \sigma_t dZ_t$.
- ▶ Thus equilibrium price of risk (and max. Sharpe ratio) is $\eta_t = \gamma \sigma_t$. (Generates equity premium puzzle given standard γ, σ_t .)

Example I: The Breeden-Lucas Economy

- ▶ Consumption Euler equation:

$$\frac{E_t(dS_t)}{S_t} - r_t dt = \gamma \text{Cov}_t \left(\frac{dS_t}{S_t}, \frac{dC_t}{C_t} \right).$$

- ▶ First, set θ and σ constant, $T = \infty$. Then:

- ▶ Risk-free rate (Itô's Lemma): $r_t = \frac{-E_t[d\pi_t]/\pi_t}{dt} = \rho + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2$ (constant).
- ▶ Constant price-dividend ratio (or wealth-consumption ratio): $W_t = \nu C_t$.
- ▶ IID returns.
- ▶ CAPM holds.

- ▶ When θ and σ are not constant:

- ▶ $W_t = \nu(t, \theta_t, \sigma_t) C_t$, so ICAPM holds.

Example II: Multiple Trees

- ▶ Following Cochrane, Longstaff, Santa-Clara (2008) and Martin (2013): Rep. agent, and i indexes individual dividend trees now. Each tree pays dividend:

$$d\delta_{i,t} = \theta_i \delta_{i,t} dt + \sigma_i \delta_{i,t} dZ_{i,t},$$

and aggregate consumption is $c_t = \sum_i \delta_{i,t}$.

- ▶ Can show an equilibrium “spillover” effect:
 - ▶ Individual trees’ dividends are IID, but expected returns, return volatility, return correlations all vary endogenously.
 - ▶ Why? Investors want to rebalance portfolio after any shock $dZ_{i,t}$; with fixed supply, prices must adjust to achieve this rebalancing. Thus expected returns and vol. vary and exhibit predictability.
 - ▶ Very small assets may comove endogenously and can earn non-zero risk premium even if their cash flows are independent of the rest of the economy.
- ▶ Applications to cross-sectional asset pricing (Menzly, Santos, Veronesi, 2004), labor income and risk premia (Santos and Veronesi, 2006).