

General Equilibrium in Complete and Incomplete Markets

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Outline

Background

GE in Complete Markets: Contingent Claims Equilibrium

GE in Incomplete Markets: Duality Approach

Application: Intermediary-Based Pricing in GE

Background

- ▶ Previously: Martingale approach in complete markets (partial equilibrium and some GE).
 - ▶ Our equilibrium notion: **securities market** equilibrium (with dynamic trade).
 - ▶ Also briefly discussed how CAPMs extended to incomplete markets.
- ▶ Last lecture: Dynamic programming approach in incomplete markets for a single agent (partial equilibrium).
 - ▶ A useful detour: Allowed us to characterize optimal trading strategies.
 - ▶ But a detour nonetheless, as we'll be returning today to the martingale approach to study GE more thoroughly. (Difficult to do GE with dynamic programming: need to guess form of price process & solve fixed-point problem.)
- ▶ Today: Putting everything together.
 - ▶ Complete markets: **contingent claims** equilibrium and the social planner's martingale problem.
 - ▶ Incomplete markets: **duality approach** to characterize GE with martingale method.

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Contingent Claims Equilibrium: Setup

Setup formally slightly different from securities market equilibrium setup:

- ▶ Start as usual with a probability space (Ω, \mathcal{F}, P) , a time interval $\mathcal{T} = [0, T]$, a Brownian motion $Z = (Z_1, \dots, Z_d)$ on (Ω, \mathcal{F}, P) , and the standard filtration \mathbb{F} of Z .

- ▶ I agents, with preferences:

$$U_i(c_i, C_{i,T}) = E \left[\int_0^T u_{i,t}(c_{i,t}) dt + U_{i,T}(C_{i,T}) \right],$$

with $u_{i,t}$ and $U_{i,T}$ strictly increasing and concave.

- ▶ Endowment: Agent i receives an endowment of the consumption good at a rate $e_i \in \mathcal{L}_+^1$, and terminal endowment $e_{i,T} \geq 0$. Write $e^i = (e_i, e_{i,T})$.
- ▶ **Contingent claims:** Arbitrary consumption plans $c^i = (c_i, C_{i,T})$ can be traded only at date 0. Price: $\Pi(c^i)$.
 - ▶ Instead of assuming N (dividend-paying) securities in positive net supply, we're assuming any state-contingent transfers are tradable at date 0 in zero net supply to achieve consumption plans.
 - ▶ But in complete markets, there's no distinction: dynamically complete securities markets \Leftrightarrow state-contingent consumption plans. Just keep in mind that aggregate endowment here is effectively equal to $e + x\delta$ in previous case.

Contingent Claims Equilibrium

Definition

A contingent claims consumption plan is **feasible** for agent i iff $\Pi(c^i) \leq \Pi(e^i)$. Denote by \mathcal{B}_{Π, e^i} the set of contingent claims feasible consumption plans for agent i .

- Price of state-contingent consumption plan can't exceed value of endowment stream.

Definition

A contingent claims consumption plan c^i is **optimal** iff it solves $\max_{c_i, C_{i,T}} U_i(c_i, C_{i,T})$ s.t. $c^i \in \mathcal{B}_{\Pi, e^i}$.

Definition (Contingent Claims Equilibrium)

A **contingent claims (CC) equilibrium** is a price function Π and set of consumption plans (c^1, \dots, c^I) , such that

1. optimality: c^i is optimal for agent i ;
2. market clearing: $\sum_{i=1}^I c^i = \sum_{i=1}^I e^i$.

Contingent Claims Equilibrium: Pricing Features

Proposition

If $\Pi: \mathcal{L}^1 \times L^1 \rightarrow \mathbb{R}$ is linear and strictly increasing, then in CC equilibrium, there is a unique and strictly positive $p \in \mathcal{L}^1$ such that

$$\Pi(c^i) = E \left[\int_0^T p_t c_{i,t} + p_T C_{i,T} \right].$$

Proposition

For any CC equilibrium, there exists an equivalent securities market equilibrium with complete markets, for which the trading strategy θ_i finances the cash flow $(-\Pi(e^i), c_i - e_i, C_{i,T})$.

Pareto Optimality

- Now set up tools to consider planner's problem and define representative agent.

Definition

A **consumption allocation** is a vector (c^1, \dots, c^I) of consumption plans.

Definition

A consumption allocation (c^1, \dots, c^I) is **feasible** iff $\sum_{i=1}^I c^i \leq \sum_{i=1}^I e^i$.

Definition

A feasible consumption allocation (c^1, \dots, c^I) is **Pareto optimal** iff there does not exist a feasible allocation $(\hat{c}^1, \dots, \hat{c}^I)$ such that $U_i(\hat{c}^i) \geq U_i(c^i)$ for all i , and $U_i(\hat{c}^i) > U_i(c^i)$ for at least one i .

First Welfare Theorem

The CC equilibrium consumption allocation is Pareto optimal.

Representative Agent

Representative Agent Problem, \mathcal{R}

For a set of **Pareto weights** $(\alpha_1, \dots, \alpha_I)$,

$$U(c) = \max_{c^1, \dots, c^I} \sum_{i=1}^I \alpha_i U_i(c^i)$$
$$\text{s.t.} \quad \sum_{i=1}^I c^i \leq c.$$

Proposition

There exist weights $(\alpha_1, \dots, \alpha_I)$ such that the CC equilibrium consumption allocation solves problem \mathcal{R} for aggregate consumption $c = e = \sum_{i=1}^I e^i$. Moreover, the equilibrium price process p and consumption plan $c = e$ is an equilibrium for the representative agent economy, and $U(c)$ inherits the time-additive expected utility form of individual utility $U_i(c^i)$.

- Typically α_i takes the form $\alpha_i = 1/\lambda_i$, where λ_i is the Lagrange multiplier on individual i 's budget constraint.

Taking Stock: Solving for GE in Complete Markets

- ▶ Consider a problem like #3 on your last problem set:
 - ▶ Aggregate endowment e_t (X_t in the problem) satisfies some SDE $de_t = \mu_t dt + \sigma_t dZ_t$.
 - ▶ There's a market where shares of the endowment are traded, and endowment is distributed in the form of a dividend flow for holding shares.
 - ▶ Different agents, with possibly different utility functions.
- ▶ If markets are complete, then you can go straight to the social planner's problem, with aggregate consumption equal to aggregate endowment:
 - ▶ Maximize utility for representative agent, $U(c) = \max_{\{c^i\}} \sum_i \alpha_i U_i(c^i)$ s.t. $\sum_i c^i = e$.
 - ▶ Just a state-by-state maximization problem! Easy to solve as a static problem given α_i .
 - ▶ Then, if needed, can figure out which weights α_i generate allocations that are feasible for the individual agents given their endowments.
- ▶ To solve for dynamics of state-price density (and risk premia, ...):
 - ▶ Can either figure out rep. agent marginal utility $\Rightarrow \pi_t = u'(c_t)/\lambda \dots$
 - ▶ ... or use the fact that individual agent optimality (from martingale method) $\Rightarrow \pi_t = u'_{i,t}(c_{i,t})/\lambda_i$ (as in CAPM slides). You'll do this in the problem set.

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GE in Incomplete Markets: Setup

- ▶ Now want to widen our scope to allow for incomplete markets (frictions, constraints, ...).
- ▶ To do so, we'll go back to the same formal setup we used for dynamic programming.
- ▶ Gains process:

$$\frac{dS_{n,t}}{S_{n,t}} = \bar{\mu}_{n,t} dt + \bar{\sigma}_{n,t} dZ_t.$$

- ▶ Given state variables X_t , assume $r_t = r(X_t, t)$, $\bar{\mu}_t = \bar{\mu}(X_t, t)$, $\bar{\sigma}_t = \bar{\sigma}(X_t, t)$, and

$$dX_t = \mu_X(X_t, t) dt + \sigma_X(X_t, t) dZ_t.$$

- ▶ Risky-asset portfolio shares for arbitrary agent: $\phi_t = (\phi_{1,t}, \dots, \phi_{N,t})$, with $\phi_{n,t} = \theta_{n,t} S_{n,t} / W_t$ (share of wealth invested in asset n).
- ▶ **Portfolio constraints:** Assume portfolio shares are restricted to lie in closed convex set $\mathbf{K} \subseteq \mathbb{R}^N$, containing the zero vector: $\phi_t \in \mathbf{K}$. Examples:
 1. No short sales: $\mathbf{K} = \{\phi : \phi \geq 0\}$.
 2. No short sales or borrowing: $\mathbf{K} = \{\phi : \phi \geq 0, \phi 1 \leq 1\}$, where $1' = (1, \dots, 1)$.

Support Functions

Definition (Support Function)

For a closed, convex set $\mathbf{K} \subseteq \mathbb{R}^N$ with $0 \in \mathbf{K}$, the **support function** of \mathbf{K} , $\delta(\cdot): \mathbb{R}^N \rightarrow \mathbb{R} \cup \infty$, is

$$\delta(v) \equiv \sup_{\phi \in \mathbf{K}} (-v' \phi), \quad v \in \mathbb{R}^N.$$

Definition (Effective Domain)

The **effective domain** of support function $\delta(v)$ is

$$\tilde{\mathbf{K}} \equiv \{v : \delta(v) < \infty\}.$$

- ▶ Support function describes boundaries of constraint set, and allows for dual characterization of optimization problem.
- ▶ Convex $\mathbf{K} \implies$ continuous $\delta(v)$, and $0 \in \mathbf{K} \implies \delta(v)$ is bounded below.
- ▶ By definition, $\phi \in \mathbf{K} \iff \delta(v) + v' \phi \geq 0 \quad \forall v \in \tilde{\mathbf{K}}$.
- ▶ Restrict v to the set $\mathbf{D} \equiv \left\{ v_t, 0 \leq t \leq T : v_t \in \tilde{\mathbf{K}}, E_0 \left[\int_0^T \delta(v_t) dt \right] + E_0 \left[\int_0^T \|v_t\|^2 dt \right] < \infty \right\}$.

Support Functions

Definition

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$$\delta(v) \equiv \sup_{\phi \in \mathbf{K}} (-v' \phi), \quad v \in \mathbb{R}^N,$$

and its **effective domain** is $\tilde{\mathbf{K}} \equiv \{v : \delta(v) < \infty\}$.

Examples:

1. No constraint: $\mathbf{K} = \mathbb{R}^N \implies \tilde{\mathbf{K}} = \{0\}$ and $\delta(v) = 0$ on $\tilde{\mathbf{K}}$ (and ∞ otherwise).
2. No short sales: $\mathbf{K} = [0, \infty)^N \implies \tilde{\mathbf{K}} = \mathbf{K}$ and $\delta(v) = 0$ on $\tilde{\mathbf{K}}$ (and ∞ otherwise).
3. Incomplete markets: Model (WLOG) by assuming N Brownian processes and N securities ($\text{rank}(\sigma_X) = \text{rank}(\bar{\sigma}_t) = N$), but only the first L securities can be traded, and positions are restricted to 0 for the rest. Then $\mathbf{K} = \{\phi : \phi_i = 0 \text{ for } L < i \leq N\} \implies \delta(v) = 0$ if $v_i = 0$ for $1 \leq i \leq L$ (and ∞ otherwise), so $\tilde{\mathbf{K}} = \{v : v_i = 0 \text{ for } 1 \leq i \leq L\}$.
4. Incomplete markets, no short sales: $\mathbf{K} = \{\phi : \phi \geq 0, \phi_i = 0 \text{ for } L < i \leq N\} \implies \delta(v) = 0$ if $v_i \geq 0, i = 1, \dots, L$ (∞ otherwise).
5. Incomplete markets, no short sales, no borrowing: $\mathbf{K} = \{\phi : \phi \geq 0, \phi_1 \leq 1, \phi_i = 0 \text{ for } L < i \leq N\} \implies \delta(v) = \max(0, -v_1, \dots, -v_L)$ (finite for any $v > -\infty$, so $\tilde{\mathbf{K}} = \mathbb{R}^N$).

Support Functions

Definition

For a closed, convex set $\mathbf{K} \subseteq \mathbb{R}^N$ with $0 \in \mathbf{K}$, the **support function** of \mathbf{K} , $\delta(\cdot) : \mathbb{R}^N \rightarrow \mathbb{R} \cup \infty$, is

$$\delta(v) \equiv \sup_{\phi \in \mathbf{K}} (-v' \phi), \quad v \in \mathbb{R}^N,$$

and its **effective domain** is $\tilde{\mathbf{K}} \equiv \{v : \delta(v) < \infty\}$.

Why are these useful?

- ▶ For incomplete market described by arbitrary \mathbf{K} , we're going to look for a fictitious complete market with (i) a higher risk-free rate, (ii) different price of risk, and (iii) different asset returns, such that an *unconstrained* investor would optimally choose a portfolio satisfying the constraints \mathbf{K} .
- ▶ The compensation required (in the form of (i)+(ii)+(iii)) to get an unconstrained investor to choose $\phi \in \mathbf{K}$ then give us the shadow prices of those constraints in the (non-fictitious) market.
- ▶ There will typically be multiple fictitious markets in which this is the case. We look for a fictitious market described by a v such that the unconstrained investor's value function is equal to the constrained investor's value function in the non-fictitious market.
- ▶ It turns out that this is equivalent to requiring a complementary slackness condition.

Dual Problem

- Constraints are \mathbf{K} and support function is $\delta(v) \equiv \sup_{\phi \in \mathbf{K}} (-v' \phi)$.
- For each $v \in \mathbf{D}$, define fictitious market $M^{(v)}$ without constraints, but with risk-free rate process and expected returns given by:

$$r_t^{(v)} = r_t + \delta(v_t),$$

$$\bar{\mu}_t^{(v)} = \bar{\mu}_t + \delta(v_t) + v_t.$$

- Recall that $\delta(v)$ is bounded below by 0, so the risk-free rate is (weakly) higher in the v economy.
- Excess returns depend on v . In, e.g., incomplete markets, no short sales, no borrowing economy, we'll need a v with negative entries (will become clear why), so excess returns in fictitious economy are lower (higher risk-free rate but lower risk premia needed to get agent to optimally hold positive, unlevered positions).
- Second condition above is equivalent to specifying price-of-risk process ~~$\eta_t^{(v)} = \eta_t + \bar{\sigma}_t^{-1} \eta_t$~~

$$\eta_t^{(v)} = \eta_t + \bar{\sigma}_t^{-1} v_t,$$

since expected returns are ~~$\bar{\mu}_t^{(v)} = r_t^{(v)} + \bar{\sigma}_t + \eta_t^{(v)}$~~ $\bar{\mu}_t^{(v)} = r_t^{(v)} + \bar{\sigma}_t \eta_t^{(v)}$. [Corrected 11/17]

Dual Problem

- Constraints are \mathbf{K} and support function is $\delta(v) \equiv \sup_{\phi \in \mathbf{K}} (-v' \phi)$. For each $v \in \mathbf{D}$, define fictitious market $M^{(v)}$ without constraints, but with risk-free rate process and expected returns given by:

$$\begin{aligned} r_t^{(v)} &= r_t + \delta(v_t), \\ \bar{\mu}_t^{(v)} &= \bar{\mu}_t + \delta(v_t) + v_t. \end{aligned}$$

- Portfolio choice in fictitious market: no constraints, so can use static (martingale) method:

$$V^{(v)} \equiv \sup_{c_t, W_T} E_0 \left[\int_0^T u_t(c_t) dt + U(W_T) \right] \quad \text{s.t.} \quad E_0 \left[\pi_T^{(v)} W_T \right] \leq W_0, \quad (\mathcal{P}^{(v)})$$

with $\pi_t^{(v)} = \exp \left(- \int_0^t r_s^{(v)} ds - \frac{1}{2} \int_0^t \eta_s^{(v)'} \eta_s^{(v)} ds - \int_0^t \eta_s^{(v)'} dZ_s \right)$ and $\eta_t^{(v)} = \eta_t + \bar{\sigma}_t^{-1} v_t$.

- Portfolio value in fictitious market: $dW_t^{(v)} = W_t^{(v)} \left[\left(r_t^{(v)} + \phi_t \bar{\sigma}_t \eta_t^{(v)} \right) dt + \phi_t \bar{\sigma}_t dZ_t \right]$. Thus:

$$\frac{dW_t^{(v)}}{W_t^{(v)}} - \frac{dW_t}{W_t} = \left[\left(r_t^{(v)} - r_t \right) + \phi_t \bar{\sigma}_t \left(\eta_t^{(v)} - \eta_t \right) \right] dt = \left[\delta(v_t) + \phi_t v_t \right] dt \geq 0,$$

where the inequality uses $\delta(v) + v' \phi \geq 0$. We therefore have $V_0^{(v)} \geq V_0$, with equality if $\delta(v_t) + \phi_t v_t = 0$.

- Thus look for v under which $\delta(v_t) + \phi_t v_t = 0$ and $\phi_t^{(v)} \in \mathbf{K}$, so that the strategy is feasible in the original (constrained) market. Then we've found a solution to the constrained problem using martingale method in unconstrained v market. (See Cvitanic and Karatzas (1992) for technical conditions.) This is like minimizing $V_0^{(v)}$ w.r.t. v s.t. $W_t^{(v)}$ is feasible in original problem.

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Intermediary-Based Pricing: Basic Model

- ▶ Based on Brunnermeier and Sannikov (2016), building on Basak and Cuoco (1998) model of (exogenously) limited stock market participation.
- ▶ Exchange economy. One stock. Risk-free bond in zero net supply.
- ▶ Dividend on the stock:

$$d\delta_t = \mu\delta_t dt + \sigma\delta_t dZ_t.$$

- ▶ 2 types of agents: Experts (intermediaries) can invest in risky asset by borrowing risk-free from households, but can't issue equity. Equivalent to assuming experts can trade in both stocks and bonds, while households can only trade in bonds. (Can weaken this.)
- ▶ Experts referred to as type 1, households are type 2. Utility:

$$U_1 = E_0 \left[\int_0^T e^{-\rho t} u(c_{1,t}) dt \right], \quad U_2 = E_0 \left[\int_0^T e^{-\rho t} \ln(c_{2,t}) dt \right].$$

- ▶ Initial endowments:
 - ▶ At time $t = 0$, agent 2 is endowed with A units of the bond, priced at 1.
 - ▶ Agent 1 is endowed with one share of the stock, is short A shares of the bond.

Individual Optimization

- ▶ Consider the constrained agent's dual problem.
- ▶ Market price of risk in fictitious market $\eta^{(v)}$, interest rate $r^{(v)} = r + \delta(v_t)$.
- ▶ Since the constrained agent can freely access the bond market (case 3 on slide 14), support function is equal to zero $\Rightarrow r_t^{(v)} = r_t$.
- ▶ In the fictitious economy where the log agent is unconstrained, optimal portfolio is myopic: From Merton solution on slide 14 of last lecture,

$$\phi_t^{(v)} = \frac{\eta^{(v)}}{\bar{\sigma}_R}.$$

- ▶ Want $v, \eta^{(v)}$ s.t. $\phi_t^{(v)} = 0$ since in constrained economy, this agent has stock holdings of zero. Conclude that $\eta^{(v)} = 0$.
- ▶ State-price density of the constrained agent (in both fictitious and actual economy):

$$d\pi_t^{(2)} = -r_t \pi_t^{(2)} dt - 0 \pi_t^{(2)} dZ_t = -r_t \pi_t^{(2)} dt \quad \Rightarrow \quad \pi_t^{(2)} = B_t^{-1}.$$

Individual Optimization

- ▶ Denote π_t as state-price density of the unconstrained agent, η_t as market price of risk:

$$d\pi_t = -r_t\pi_t dt - \eta_t\pi_t dZ_t.$$

- ▶ Optimality conditions for both agents:

$$\begin{aligned} e^{-\rho t} u'_1(c_{1,t}) &= a_1 \pi_t, \\ e^{-\rho t} u'_2(c_{2,t}) &= a_2 B_t^{-1} \end{aligned}$$

- ▶ Define ratio of SDFs:

$$\xi_t = \frac{a_1 \pi_t}{a_2 \pi_t^{(c)}} = \frac{a_1}{a_2} \pi_t B_t.$$

- ▶ Consumption-sharing rule:

$$\frac{u'_1(c_{1,t})}{u'_2(c_{2,t})} = \xi_t, \quad c_{1,t} + c_{2,t} = \delta_t.$$

- ▶ Denote solution $c_{1,t} = F(\xi_t, \delta_t)$.

Market Clearing

- ▶ Characterize dynamics of ξ_t using market clearing conditions. First:

$$\frac{d\xi_t}{\xi_t} = \frac{d\pi_t}{\pi_t} + \frac{dB_t}{B_t} = -r_t dt - \eta_t dZ_t + r_t dt = -\eta_t dZ_t.$$

Thus, $\frac{d\xi_t}{\xi_t}$ equals the diffusion component of $\frac{d\pi_t}{\pi_t}$, and therefore

$$\frac{d\xi_t}{\xi_t} = \text{stoch} \left(\frac{u_1''(c_{1,t})}{u_1'(c_{1,t})} dc_{1,t} \right) = \frac{u_1''(F(\xi_t, \delta_t))}{u_1'(F(\xi_t, \delta_t))} \sigma \delta_t dZ_t.$$

- ▶ Last equality follows from

$$dc_{1,t} = d\delta_t - dc_{2,t} \quad \text{and} \quad \text{stoch}(dc_{2,t}) = 0.$$

- ▶ Just need initial value ξ_0 to completely characterize equilibrium allocations.
- ▶ Then can compute prices using SPD

$$\pi_t = e^{-\rho t} \frac{u_1'(c_{1,t})}{u_1'(c_{1,0})}.$$

Equilibrium

- ▶ To solve for ξ_0 , use budget constraint of agent 2.
- ▶ Optimal consumption policy for log agent (solving ODE from dynamic programming problem) gives

$$c_{2,0} = W_{2,0} \left(\frac{1 - e^{-\rho T}}{\rho} \right)^{-1}, \quad W_{2,0} = A.$$

- ▶ Using definition of ξ_t , have $\xi_t = u'_1(c_{1,t})c_{2,t}$.
- ▶ ξ_0 must solve

$$\frac{\xi_0}{u'_1(F(\xi_0, \delta_0))} \frac{1 - e^{-\rho T}}{\rho} = A.$$

Intermediary-Based Pricing: Intuition

- ▶ Note that even without solving for ξ_0 , can see from market clearing slide that limited participation increases market price of risk:

$$\eta_t = - \frac{c_{1,t} u_1''(c_{1,t})}{u_1'(c_{1,t})} \frac{\delta_t}{c_{1,t}} \sigma.$$

- ▶ Negative dividend shocks are shocks to the intermediary's net worth, since the intermediary holds all the equity risk.
- ▶ Thus intermediary net worth shocks (increases in $\delta_t / c_{1,t}$, which is 1 + type-2 consumption share) increase the market price of risk and thus the risk premium on equity.
- ▶ The intermediary has to delever in the face of a negative shock, but since it has to be able to afford equity, the price of equity declines and risk premium rises.

Discussion

- ▶ It is common to interpret ξ_t as the stochastic utility weight for the constrained agent. If we solved

$$\sup u_1(c_1) + \xi u_2(c_2) \quad \text{s.t. } c_1 + c_2 = \delta,$$

would recover the same consumption allocation.

- ▶ Thus like a planning problem with Pareto weights ξ_t .
- ▶ Could solve for ξ_t instead of searching for price processes directly.
- ▶ High volatility of ξ_t implies high volatility of SDF.
- ▶ Under complete markets, ξ_t is constant.

Log Utility Solution

- Assume agent 1 also has log utility, so

$$c_2 = \frac{\xi}{\xi + 1} \delta, \quad c_1 = \frac{\delta}{\xi + 1}.$$

- Evolution of ξ_t is given by

$$d\xi_t = -\xi_t \sigma \frac{\delta_t}{c_{1,t}} dZ_t = -\xi_t (\xi_t + 1) \sigma dZ_t.$$

- Initial condition satisfies

$$\xi_0 \frac{\delta_0}{1 + \xi_0} \frac{1 - e^{-\rho T}}{\rho} = A \quad \Rightarrow \quad \xi_0 = \frac{\rho A}{\delta_0(1 - e^{-\rho T}) - \rho A}.$$

- Solve for ξ_0 :

$$S_0 = \frac{1 - e^{-\rho T}}{\rho} \delta_0 \quad \Rightarrow \quad \xi_0 = \frac{A}{S_0 - A} = \frac{W_{2,0}}{W_{1,0}}$$

- At $t = 0$,

$$\eta_0 = (1 + \xi_0) \sigma \quad \nearrow \quad \text{in} \quad \frac{W_{2,0}}{W_{1,0}}$$

Log Utility Solution

- Risk-free rate

$$\pi_t = e^{-\rho t} \frac{1 + \xi_t}{\delta_t}, \quad E_t \left[-\frac{d\pi_t}{\pi_t} \right] = r_t dt$$

$$r_0 = \rho + \mu - (1 + \xi_0)\sigma^2$$

- For large enough $\frac{W_{2,0}}{W_{1,0}}$, obtain a high market price of risk and a low risk-free rate.
- Volatility of stock returns is still equal to the volatility of dividend growth.

Adding Production

- ▶ Now assume risky asset is a claim on production from physical capital, with AK technology and capital quality shocks. Output is $(a - \iota_t)k_t dt$, with ι_t the rate of investment.
- ▶ Capital stock evolves as

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta) dt + \sigma dZ_t,$$

with $\Phi(0) = 0, \Phi' > 0, \Phi'' \leq 0$.

- ▶ Capital return:

$$dr_t^k = \frac{a - \iota_t}{q_t} dt + (\Phi(\iota_t) - \delta + \mu_t^q + \sigma \sigma_t^q) dt + (\sigma + \sigma_t^q) dZ_t,$$

where first term is dividend yield and remaining terms are capital gains.

- ▶ Optimal investment requires $\Phi' = 1/q_t$, and remainder of equilibrium conditions are the same as before. Gives price of risk process $\frac{\rho + \Phi(\iota) - \delta - r_t}{\sigma}$, which is proportional to the inverse of the intermediary's wealth share.
- ▶ You'll see some variants of intermediary-based models with Kerry after Thanksgiving.