

# Dynamic Programming

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# Outline

Background

Dynamic Programming in Continuous Time

Application (Incomplete Markets, Partial Equilibrium): Portfolio Choice with Margin Constraints

Stochastic Differential Utility

# Background

- ▶ Last class, we considered the martingale approach to portfolio and consumption choice.
  - ▶ “Martingale approach”  $\Leftrightarrow$  discounted consumption is a martingale under  $\mathbb{Q}$ ; since  $\mathbb{Q}$  is unique in complete markets, can characterize optimal state-contingent *payoffs* (i.e., optimal consumption choice) from static problem, then later figure out portfolio choice to implement these payoffs.
- ▶ Then considered equilibrium and CAPM-type equations.
  - ▶ Discussed how CAPM equations generalize to incomplete markets, but many of our derivations used complete markets and martingale method.
  - ▶ We considered *securities market equilibrium* (with sequential trading). As an aside, in complete markets one can equivalently define *contingent claims equilibrium* (with Arrow-Debreu claims traded only at date 0) and show welfare theorems and existence of rep. agent.
- ▶ In incomplete markets, can solve choice problems with martingale approach and use duality result (next lecture). But useful to go over the alternative approach: **dynamic programming**.
  - ▶ Often easier to solve, give cleaner interpretations for portfolio choice rules.
  - ▶ Downside: GE less straightforward.

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# Dynamic Programming Setup

- Price process (notation as in second half of previous lecture):

$$\frac{dS_{n,t}}{S_{n,t}} = \bar{\mu}_{n,t} dt + \bar{\sigma}_{n,t} dZ_t.$$

- Assume short rate, price drift, and diffusion depend only on state variables  $X_t$  and time,  $r_t = r(X_t, t)$ ,  $\bar{\mu}_t = \bar{\mu}(X_t, t)$ ,  $\bar{\sigma}_t = \bar{\sigma}(X_t, t)$ , and the state vector is a Markov process satisfying SDE:

$$dX_t = \mu_X(X_t, t) dt + \sigma_X(X_t, t) dZ_t.$$

- Investor's wealth at time  $t$ :  $W_t = \theta_{0,t}B_t + \theta_{1N,t}S_t$ . The investor chooses “controls” at time  $t$ ,  $(c_t, \theta_{1N,t})$ , assumed to be a function of the state at  $t$ .
- Define  $\phi_{n,t} = \theta_{n,t}S_{n,t}/W_t$  (share of wealth invested in asset  $n$ ),  $\phi_t = (\phi_{1,t}, \dots, \phi_{N,t})$ , rewrite controls as  $(c_t, \phi_t)$ .

# Dynamic Programming Setup

- ▶ Given wealth  $W_t = \theta_{0,t}B_t + \theta_{1N,t}S_t$  and controls, wealth evolves according to SDE:

$$\begin{aligned}dW_t &= \theta_{0,t}dB_t + \theta_{1N,t}dS_t - c_t dt \\ &= (\theta_{0,t}r_t B_t + \theta_{1N,t}\mu_t - c_t)dt + \theta_{1N,t}\sigma_t dZ_t,\end{aligned}$$

where  $\mu_t = I_{S_t}\bar{\mu}_t$ ,  $\sigma_t = I_{S_t}\bar{\sigma}_t$ . (First line follows from definition of self-financing strategy.)

- ▶ Given definitions  $\phi_{n,t} = \theta_{n,t}S_{n,t}/W_t$  and  $\phi_t = (\phi_{1,t}, \dots, \phi_{N,t})$ , rewrite this dynamic budget constraint as:

$$\begin{aligned}dW_t &= (W_t r_t + W_t \phi_t (\bar{\mu}_t - r_t 1) - c_t) dt + W_t \phi_t \bar{\sigma}_t dZ_t \\ &= (W_t r(X_t, t) + W_t \phi_t (\bar{\mu}(X_t, t) - r(X_t, t) 1) - c_t) dt + W_t \phi_t \bar{\sigma}(X_t, t) dZ_t.\end{aligned}$$

# Dynamic Programming Problem

## Investor's Problem, $\mathcal{P}_t$

The investor's problem at time  $t$  is

$$\max_{c_s, \phi_s} E_t \left[ \int_t^T u_s(c_s) ds + U_T(C_T) \right],$$

subject to

$$dW_s = \left( W_s r(X_s, s) + W_s \phi_s (\bar{\mu}(X_s, s) - r(X_s, s)) - c_s \right) ds + W_s \phi_s \bar{\sigma}(X_s, s) dZ_s,$$

$$dX_s = \mu_X(X_s, s) ds + \sigma_X(X_s, s) dZ_s,$$

and  $C_T = W_T$ . Denote the supremum of  $\mathcal{P}_t$  by the value function  $V(t, W_t, X_t)$ .

Note that the choice is over controls for  $s \in [t, T]$ . We're going to derive a continuous-time Bellman equation (HJB equation) that turns it into a sequential program over controls at  $t$ .

## A Heuristic Derivation of the HJB Equation

- Consider a discrete-time world with time interval  $\Delta t$  corresponding to a period. One-period consumption is  $c_t \Delta t$ , and per-period utility is  $u_t(c_t) \Delta t$ . The one-period bond return is  $r_t \Delta t$ , and one-period stock returns are

$$S_{t+\Delta t}/S_t = 1 + \bar{\mu}_t \Delta t + \bar{\sigma}_t \sqrt{\Delta t} \epsilon_t,$$

where i.i.d. shocks  $\epsilon_t = \pm 1$  with equal probability.

- Assume for now a one-dimensional state  $X_t$ . The Bellman equation is

$$J(t, W_t, X_t) = \max_{c_t, \phi_t} [u_t(c_t) \Delta t + E_t[J(t + \Delta t, W_{t+\Delta t}, X_{t+\Delta t})]],$$

where

$$X_{t+\Delta t} = X_t + \mu_{X,t} \Delta t + \sigma_{X,t} \sqrt{\Delta t} \epsilon_t,$$

$$W_{t+\Delta t} = W_t + (W_t r_t \Delta t + W_t \phi_t (\bar{\mu}_t - r_t) \Delta t - c_t \Delta t) + W_t \phi_t \bar{\sigma}_t \sqrt{\Delta t} \epsilon_t.$$



# A Heuristic Derivation of the HJB Equation

$$J(t, W_t, X_t) = \max_{c_t, \phi_t} [u_t(c_t)\Delta t + E_t[J(t + \Delta t, W_{t+\Delta t}, X_{t+\Delta t})]],$$

$$X_{t+\Delta t} = X_t + \mu_{X,t} \Delta t + \sigma_{X,t} \sqrt{\Delta t} \epsilon_t,$$

$$W_{t+\Delta t} = W_t + (W_t r_t \Delta t + W_t \phi_t (\bar{\mu}_t - r_t) \Delta t - c_t \Delta t) + W_t \phi_t \bar{\sigma}_t \sqrt{\Delta t} \epsilon_t.$$

- Take Taylor expansion of expectation term around  $t, W_t, X_t$  (using subscripts for partial derivatives):

$$E_t[J(t + \Delta t, W_{t+\Delta t}, X_{t+\Delta t})] = E_t \left[ \begin{aligned} &J(t, W_t, X_t) + J_t \Delta t + J_W(W_{t+\Delta t} - W_t) + J_X(X_{t+\Delta t} - X_t) \\ &+ \frac{1}{2} (J_{WW}(W_{t+\Delta t} - W_t)^2 + J_{XX}(X_{t+\Delta t} - X_t)^2) \\ &+ J_{WX}(W_{t+\Delta t} - W_t)(X_{t+\Delta t} - X_t) \end{aligned} \right] + o(\Delta t).$$

- Take  $\Delta t \rightarrow 0$ , keep terms of order  $t$ , and get PDE for  $J$ . ( $E_t[J_X(X_{t+\Delta t} - X_t)]$  becomes  $J_X \mu_X$ ,  $E_t[(1/2)J_{WW}(W_{t+\Delta t} - W_t)^2]$  becomes  $(1/2)W^2 \phi_t \bar{\sigma}_t \bar{\sigma}_t' \phi_t'$ , ...) Resulting PDE is the continuous-time Bellman equation.

# The Bellman Equation

## Theorem

Suppose that a twice continuously differentiable function  $V(t, W_t, X_t)$  is the value function and that a control  $(c_t^*, \phi_t^*)$  is optimal for the problem  $\mathcal{P}_t$ . Then  $V(t, W_t, X_t)$  solves the Bellman equation

$$\max_{c_t, \phi_t} \left[ u_t(c_t) + \mathcal{D}_{WX}^{c\phi} V(t, W_t, X_t) + V_t(t, W_t, X_t) \right] = 0, \quad (\star)$$

with the terminal condition  $V(T, W_T, X_T) = U_T(W_T)$ , where

$$\begin{aligned} \mathcal{D}_{WX}^{c\phi} V = & V_W (W_t r_t + W_t \phi_t (\bar{\mu}_t - r_t 1) - c_t) + V'_X \mu_{X,t} \\ & + \frac{1}{2} \left( W_t^2 V_{WW} \phi_t \bar{\sigma}_t \bar{\sigma}'_t \phi'_t + 2 W_t V'_{WX} \sigma_{X,t} \bar{\sigma}'_t \phi'_t + \text{tr}(\sigma_{X,t} \sigma'_{X,t} V_{XX}) \right). \end{aligned}$$

Moreover, the maximum in the Bellman equation is achieved for the control  $(c_t^*, \phi_t^*)$ .

Useful heuristic version:

$$\max_{c_t, \phi_t} \left( u_t(c_t) dt + E_t[dV] \right) = 0.$$

Value function is max. discounted utility from  $t$  on, so consumption utility has to offset change in value.

# The Bellman Equation

Previous theorem tells us Bellman equation is necessary (if  $V$  is the value function, then it solves Bellman equation). The other, sufficient, direction holds too (if  $V$  solves Bellman eq'n, then it's the value function).

## Theorem (Verification Theorem)

Suppose that a twice continuously differentiable function  $V(t, W_t, X_t)$  solves the Bellman equation ( $\star$ ) with the terminal condition  $V(T, W_T, X_T) = U_T(W_T)$ . Suppose also that the maximum in the Bellman equation is achieved for a control  $(c_t^*, \phi_t^*)$ . Suppose finally that for all controls we have

$$E \left[ \int_0^T (V_W(t, W_t, X_t) W_t \phi_t \bar{\sigma}_t)^2 dt \right] + E \left[ \int_0^T (V'_X(t, W_t, X_t) \sigma_{X,t})^2 dt \right] < \infty.$$

Then the function  $V(t, W_t, X_t)$  is the value function, and the control  $(c_t^*, \phi_t^*)$  is optimal.

- ▶ The condition on controls is important (ensures the relevant process is a martingale) but sometimes non-trivial to verify.
- ▶ Previous two results: Can determine value function by solving Bellman equation ( $\max_{c_t, \phi_t} (u_t(c_t) dt + E_t[dV]) = 0$ ) with terminal condition  $V(T, W_T, X_T) = U_T(W_T)$ .

# Solving for the Optimal Control and Value Function

- ▶ Previous results give us steps to solve for optimal portfolio choice and the value function:
  1. Use Bellman equation to solve for optimal control as a function of partial derivatives of the value function. This characterization is sometimes (but not usually) sufficient for question at hand.
  2. Plug the optimal control back into the Bellman equation to obtain a PDE for the value function.
  3. Solve this PDE with terminal condition  $V(T, W_T, X_T) = U_T(W_T)$ .
- ▶ Optimal control is the solution to

$$\max_{c_t, \phi_t} \left[ u_t(c_t) + \mathcal{D}_{WX}^{c\phi} V(t, W_t, X_t) + V_t(t, W_t, X_t) \right],$$

where

$$\begin{aligned} \mathcal{D}_{WX}^{c\phi} V = & V_W (W_t r_t + W_t \phi_t (\bar{\mu}_t - r_t 1) - c_t) + V'_X \mu_{X,t} \\ & + \frac{1}{2} \left( W_t^2 V_{WW} \phi_t \bar{\sigma}_t \bar{\sigma}_t' \phi_t' + 2 W_t V'_{WX} \sigma_{X,t} \bar{\sigma}_t' \phi_t' + tr(\sigma_{X,t} \sigma_{X,t}' V_{XX}) \right). \end{aligned}$$

- ▶ The first-order condition for consumption (step 1(a)) gives us envelope condition:

$$u_t'(c_t^*) = V_W(t, W_t, X_t). \quad (*)$$

# Optimal Portfolio Rule

- ▶ The first-order condition for the choice of risky asset shares (step 1(b)) is

$$V_W W_t (\bar{\mu}_t - r_t \mathbf{1}) + W_t^2 V_{WW} \bar{\sigma}_t \bar{\sigma}_t' (\phi_t^*)' + W_t \bar{\sigma}_t \sigma_{X,t}' V_{WX} = 0.$$

- ▶ Assuming that the matrix  $\bar{\sigma}_t \bar{\sigma}_t'$  is invertible, we get

$$(\phi_t^*)' = (\bar{\sigma}_t \bar{\sigma}_t')^{-1} \left[ -\frac{V_W}{W_t V_{WW}} (\bar{\mu}_t - r_t \mathbf{1}) - \frac{1}{W_t V_{WW}} \bar{\sigma}_t \sigma_{X,t}' V_{WX} \right]. \quad (\dagger)$$

- ▶ Optimal risky asset portfolio is thus the sum of two portfolios (two-fund theorem of Merton (1971)):
  1. **Mean-variance efficient portfolio**, with weight  $-V_W/(W_t V_{WW})$  (the inverse of the coefficient of relative risk aversion of the value function).
  2. **Hedging portfolio**. Hedging demand for asset  $n$  is large when changes in that asset's price have a big impact on the investor's marginal utility of wealth. (Classic example: changes in investment opportunity set.)
- ▶ Can then plug these solutions into the Bellman equation and do steps 2 and 3.

## A Special Case

- ▶ Now assume (i) a **constant investment opportunity set** ( $\bar{\mu}$  and  $\bar{\sigma}$ ), and (ii) **CRRA utility** ( $u_t(c_t) = bu(c)e^{-\beta t}$  and  $U_T(C_T) = u(c)e^{-\beta T}$ , where  $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ , or  $u(c) = \log(c)$  for  $\gamma = 1$ ).
- ▶ Since the investment opportunity set is constant ( $dS_t/S_t$  is i.i.d.), the value function depends only on  $W_t$  and  $t$ . Guess a value function with the same form as the utility function ( $\gamma \neq 1$ ):

$$V(t, W_t) = A_t e^{-\beta t} \frac{W_t^{1-\gamma}}{1-\gamma}.$$

- ▶ Then, from (\*) and (†), we have

$$c_t^* = \left(\frac{A_t}{b}\right)^{-\frac{1}{\gamma}} W_t, \quad (\phi^*)' = \frac{1}{\gamma} (\bar{\sigma}\bar{\sigma}')^{-1} (\bar{\mu} - r1).$$

- ▶ With a single risky asset, we get the classic Merton (1969) solution:  $\phi^* = \frac{\bar{\mu} - r}{\gamma \bar{\sigma}^2}$ .
- ▶ Plug the optimal control into the HJB equation, simplify, and we get the ODE

$$b^{\frac{1}{\gamma}} \gamma A_t^{1-\frac{1}{\gamma}} + \psi \gamma A_t + \frac{dA_t}{dt} = 0,$$

where  $\psi \equiv f(\text{parameters})$ . This + terminal cond.  $A_T = 1 \implies$  solution  $A_t = \left[ e^{\psi(T-t)} \left( 1 + \frac{b^{\frac{1}{\gamma}}}{\psi} \right) - \frac{b^{\frac{1}{\gamma}}}{\psi} \right]^\gamma$ .

## Comparison with the Martingale Approach

- ▶ Advantage of dynamic programming: doesn't make use of (thus doesn't require) market completeness.
- ▶ Disadvantage: produces a nonlinear PDE for the value function, whereas the martingale approach produces a linear PDE for the wealth function  $F(\pi_t, S_t, t)$ .
- ▶ The link between the two approaches:

$$V_W(F(\pi_t, S_t, t), S_t, t) = u'_t(c_t^*) = \lambda \pi_t.$$

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**Application (Incomplete Markets, Partial Equilibrium): Portfolio Choice with Margin Constraints**

Stochastic Differential Utility



## Example: Liu and Longstaff (2004)

- Consider a market with two assets. One is the risk-free bond, paying a constant interest rate  $r$ . The second is the risky asset, the “arbitrage opportunity,” with price process following a **Brownian bridge**:

$$dS_t = -\frac{S_t}{T-t} dt + \sigma dZ_t.$$

- Think of  $S_t$  as the cost of a convergence trade, which is guaranteed to have a terminal payoff of  $S_T = 0$  (as  $t \rightarrow T$ , drift is  $+\infty$  if  $S_t < 0$  and vice versa, guaranteeing  $S_t \rightarrow 0$  a.s. as  $t \rightarrow T$ ). Can be equivalently defined as a Brownian motion conditional on terminal value being equal to 0.
- Margin requirement: For every unit of the arbitrage trade (long or short), the trader must have  $\lambda$  dollars invested in the risk-free asset on top of the proceeds from the trade. That is, given  $\theta_t$  units of the arb. trade and  $\alpha_t$  bonds,  $\alpha_t B_t \geq |\theta_t S_t| + \lambda |\theta_t|$ . Equivalently,

$$W_t \geq \lambda |\theta_t|, \quad \lambda > 0.$$

- Assume log utility over terminal wealth, so investor's problem is

$$\begin{aligned} & \max_{\theta_t} E_0[\ln(W_T)] \\ \text{s.t.} \quad & dW_t = (W_t - \theta_t S_t) r dt + \theta_t dS_t, \quad W_t \geq \lambda |\theta_t|. \end{aligned}$$

## Example: Liu and Longstaff (2004)

- Define  $n_t = \theta_t / W_t$ , and let  $J(W_t, S_t, t)$  be the value function. Then  $J$  satisfies

$$\begin{aligned} \max_{|n_t| \leq \lambda^{-1}} J_t + J_W W_t \left( r(1 - n_t S_t) - n_t \frac{S_t}{T-t} \right) + \frac{1}{2} J_{WW} W_t^2 n_t^2 \sigma^2 \\ + J_{WS} W_t n_t \sigma^2 + J_S \left( -\frac{S_t}{T-t} \right) + \frac{1}{2} J_{SS} \sigma^2 = 0, \end{aligned}$$

with boundary condition  $J(W_T, S_T, T) = \ln(W_T)$ .

- Conjecture that  $J$  has **separable** functional form:  $J(W_t, S_t, t) = \ln(W_t) + H(S_t, t)$ .
  - To see: Use homogeneity of utility & constraint in  $W$  to conjecture that  $\theta_t = n(S_t, t) W_t$  (so  $n_t$  depends only on  $S_t$  and  $t$ ). Plugging into expected util. yields separable functional form as given.
- Then the Bellman equation reduces to

$$\max_{|n_t| \leq \lambda^{-1}} H_t + \left( r(1 - n_t S_t) - n_t \frac{S_t}{T-t} \right) - \frac{1}{2} n_t^2 \sigma^2 + H_S \left( -\frac{S_t}{T-t} \right) + \frac{1}{2} H_{SS} \sigma^2 = 0,$$

with boundary condition  $H(S_T, T) = 0$ .

- Take FOC w.r.t.  $n_t$  to get optimal portfolio choice rule; if  $|n_t| > \lambda^{-1}$  according to this condition, then set  $n_t^* = \pm \lambda^{-1}$  for  $S_t \gtrless 0$  (corner solution). Don't have to solve for  $H$  to determine this optimal strategy.

## Example: Liu and Longstaff (2004)

- Optimal portfolio strategy:

$$n_t^* = \begin{cases} -S_t \frac{r+(T-t)^{-1}}{\sigma^2}, & \left| S_t \frac{r+(T-t)^{-1}}{\sigma^2} \right| \leq \lambda^{-1}, \\ \lambda^{-1}, & -S_t \frac{r+(T-t)^{-1}}{\sigma^2} \geq \lambda^{-1}, \\ -\lambda^{-1}, & -S_t \frac{r+(T-t)^{-1}}{\sigma^2} \leq -\lambda^{-1}. \end{cases}$$

- To make this rigorous, must solve for  $H$  to confirm the value function is finite (true as long as  $\lambda > 0$ ).
- Important property of  $n_t^*$ : when  $\left| S_t \frac{r+(T-t)^{-1}}{\sigma^2} \right| \leq \lambda^{-1}$ , the trader could invest more in the arbitrage (up to the margin limit), but chooses not to!
- This is because it's possible that the arbitrage opportunity will widen and the trader will be forced to liquidate part of the position at the worst possible time (when the arbitrage opportunity is at its best).
- Quote from an anonymous trader in the paper: "So there's an arbitrage. So what? This desk has lost a lot of money on arbitrages. Arbitrages aren't particularly great trades." Describes LTCM failure well!

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# Stochastic Differential Utility

- ▶ Last lecture: Standard preferences  $\Rightarrow$  usual puzzles (equity premium, volatility, risk-free rate).
- ▶ In discrete time, often use **recursive preferences** (Epstein-Zin utility) to help address these.
- ▶ Continuous-time analogue: **stochastic differential utility** (Duffie and Epstein, 1992). Heuristic derivation starting from discrete-time E-Z:

$$J_t = \left( \left(1 - e^{-\rho dt}\right) c_t^{1-\frac{1}{\psi}} + e^{-\rho dt} E_t \left[ J_{t+dt}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right)^{\frac{1}{1-\frac{1}{\psi}}}, \quad J_T = 0$$

$$\Leftrightarrow g(U_t) = \left(1 - e^{-\rho dt}\right) \frac{c_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + e^{-\rho dt} g(E_t[U_{t+dt}]), \quad g(x) \equiv \frac{[(1-\gamma)x]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}}{1-\frac{1}{\psi}}, \quad U_t \equiv \frac{J_t^{1-\gamma}}{1-\gamma}$$

- ▶ Taylor expand in  $dt$  and keep only first-order terms:

$$\rho \frac{c_t^{1-1/\psi}}{1-1/\psi} dt - \rho g(U_t) dt + g'(U_t) E_t[dU_t] = 0 \quad \Rightarrow \quad E_t[dU_t] = - \frac{\rho \frac{c_t^{1-1/\psi}}{1-1/\psi} - \rho g(U_t)}{g'(U_t)} dt \equiv -f(c_t, U_t) dt$$

$$\Rightarrow \quad U_t = E_t \left[ \int_t^T f(c_s, U_s) ds \right], \quad U_T = 0, \quad f(c, U) = \frac{1}{1-1/\psi} \left\{ \frac{\rho c^{1-1/\psi}}{[(1-\gamma)U]^{\frac{\gamma-1/\psi}{1-\gamma}}} - \rho(1-\gamma)U \right\}$$

# Stochastic Differential Utility

$$U_t = E_t \left[ \int_t^T f(c_s, U_s) ds \right], \quad U_T = 0,$$

$$\text{with } f(c, U) = \frac{1}{1 - 1/\psi} \left\{ \frac{\rho c^{1-1/\psi}}{[(1-\gamma)U]^{\frac{\gamma-1/\psi}{1-\gamma}}} - \rho(1-\gamma)U \right\}$$

- ▶ Standard time-separable utility:  $f(c, U) = u(c) - \rho U$ .
- ▶ Asset pricing implications: Consider constant investment opportunity set, risk-free rate (Black-Scholes economy). Bellman equation with stochastic differential utility: Value function  $V(W_t, t)$  solves

$$\max_{c, \phi} f(c, V) + V_t + V_W [(r + (\mu_R - r)\phi)W - c] + V_{WW} \sigma_R^2 \phi^2 W^2 / 2 = 0.$$

- ▶ Guess  $V(W_t, t) = A(t) W_t^{1-\gamma}$  still works, giving portfolio composition equivalent to Merton rule:

$$\phi^* = \frac{\mu_R - r}{\gamma \sigma_R^2}.$$

- ▶ But consumption policy is different (now depends on  $\psi$  and  $\gamma$  separately).
- ▶ GE similar to discrete time; can solve with  $\pi_t = \exp \left( \int_0^t f_U(c_s, U_s) ds \right) f_c(c_t, U_t) \Rightarrow \eta_t dZ_t = -\frac{df_c(c_t, U_t)}{f_c(c_t, U_t)}$  and  $r_t = r = \rho + \frac{\mu_c}{\psi} - \frac{1}{2} \frac{1}{1+\psi^{-1}} \gamma \sigma_c^2$  (can have higher  $\gamma$  without extremely low  $r$ ).