

Applications: Incomplete Information and Learning

Updated 12/6 [Jumps, Belief Disagreement]

EBEN LAZARUS

MIT 15.472

Fall 2022

Outline

Background

Jumps

Learning with Parameter Uncertainty

Learning with Imperfectly Observable States

Belief Disagreement

Background

- ▶ This week: Assorted topics in continuous time, with some emphasis on recent work.
 - ▶ Brief detours on additional tools (jumps, filtering) along the way.
 - ▶ Can spend more/less time on things based on interest (thoughts still welcome for Thursday!).
- ▶ After quick intro to jumps, discuss two types of models with incomplete information:
 1. Uncertainty about model parameters.
 2. Imperfectly observable state variables.
 - ▶ Often isomorphic to complete-information economy with nonlinear state variable dynamics.
 - ▶ Parameter uncertainty (case 1) typically has transient effect (eventually learn parameter value).
 - ▶ But imperfect observability of time-varying state variables can generate stationary equilibrium.
- ▶ Then belief disagreement: Dogmatic differences of opinion over model parameters or outcomes.
 - ▶ Will go over a nice recent paper.

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Jumps

- ▶ Brownian motions as sources of fundamental risk \Rightarrow continuous sample paths.
 - ▶ Non-restrictive as model of continuous information (given martingale representation theorem)...
 - ▶ ...but rules out discontinuous paths.
- ▶ Empirically, some price processes seem to have occasional large jumps (announcements, disasters) that are worth modeling; also useful for learning model with regime switching.
- ▶ Start from standard probability space with process Z . Will assume Z is *right-continuous* with *left limits* (càdlàg).

Definition

A process Z is **right-continuous** if, for all t , $Z_t = \lim_{s \searrow t} Z_s$; it has **left limits** if $Z_{t-} = \lim_{s \nearrow t} Z_s$ exists.

Definition

The **jump** ΔZ of Z at time t is $\Delta Z_t = Z_t - Z_{t-}$.

- ▶ Càdlàg property \Rightarrow total # of jumps on $[0, T]$ is countable.

Semimartingales and Poisson Processes

Definition

A **semimartingale** is a process X such that $X = M + V$, where M is a local martingale and V is an adapted càdlàg process with finite total variation.

- ▶ Can WLOG take M continuous (in general infinite variation) and V “pure” jump component.

Definition

A **Poisson process** is a stochastic process N in \mathbb{R} such that

1. $N_0 = 0$ a.s.,
2. for all $s > t$, $N_s - N_t$ is Poisson-distributed with intensity λ : $\Pr(N_s - N_t = k) = \exp(-\lambda(s-t)) \frac{(\lambda(s-t))^k}{k!}$.
3. for all $0 \leq t_0 < t_1 < \dots < t_n \leq T$, the increments $N_{t_0}, N_{t_1} - N_{t_0}, \dots, N_{t_n} - N_{t_{n-1}}$, are independent.

- ▶ Like a Brownian motion, but for counting # of events N_t . Not a martingale, as $E_t[N_s - N_t] = \lambda(s-t)$.
Can define **compensated Poisson process** $Y_t = N_t - \lambda t$, which is now a martingale.

Models with Jumps: Brief Primer

- ▶ Just as Brownian motions are building blocks for continuous models, Poisson processes are building blocks for jump component of semimartingale models.
 - ▶ Can generalize from this building block: Compound Poisson processes (random jump sizes at Poisson times, as in disaster models); time-varying intensity; gamma processes (infinite # of jumps over any interval, with jump of size $\in [x, x + dx)$ occurring as Poisson process with intensity $\lambda(x)dx$).
- ▶ Most of the concepts we've covered for continuous local martingales generalize straightforwardly to semimartingales.
 - ▶ Stochastic integral now has additional term for V .
 - ▶ Itô's lemma has additional jump term.
- ▶ Instantaneous covariances between pure jump terms and Brownian terms are $o(dt)$.
 - ▶ SDF has to have jump component in order for jump in price process to affect risk premia.
- ▶ Markov chains: Take two states, $X \in \{0, 1\}$, and $dX_t = (1 - X_{t-})dN_t^0 - X_{t-}dN_t^1$, where N_t^0 and N_t^1 are independent Poisson processes. Generalizes to multiple states (more shortly).

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Parameter Uncertainty: Setup

- ▶ Complete markets, partial eq'm for single investor. Results then apply to GE as usual.
- ▶ Back to diffusion process for now. Constant interest rate r , single risky asset (stock) with constant investment opportunity set:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t.$$

- ▶ Assume true value of μ is unknown. Time-0 prior $\mu \sim \mathcal{N}(m_0, v_0)$. (What about σ ?)
- ▶ Cts-time version of Kalman filter (Liptser and Shiriyayev, 1977): Under investor's beliefs,

$$\begin{aligned}\frac{dS_t}{S_t} &= m_t dt + \sigma d\tilde{Z}_t, \\ dm_t &= \frac{v_t}{\sigma} d\tilde{Z}_t, \\ d\tilde{Z}_t &= \frac{1}{\sigma} \left(\frac{dS_t}{S_t} - m_t dt \right), \\ dv_t &= -\frac{v_t^2}{\sigma^2} dt.\end{aligned}$$

- ▶ \tilde{Z}_t is (scaled) return relative to expectation. Standard Brownian motion under investor's info set.

Parameter Uncertainty: Portfolio Choice

- ▶ True process: $\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t$.
- ▶ Under investor's info set:

$$\begin{aligned}\frac{dS_t}{S_t} &= m_t dt + \sigma d\tilde{Z}_t, & dm_t &= \frac{v_t}{\sigma} d\tilde{Z}_t, \\ d\tilde{Z}_t &= \frac{1}{\sigma} \left(\frac{dS_t}{S_t} - m_t dt \right), & dv_t &= -\frac{v_t^2}{\sigma^2} dt.\end{aligned}$$

- ▶ Notice uncertainty v_t declines deterministically (is this good or bad?).
- ▶ Stochastic market price of risk for investor: $\eta_t = (m_t - r)/\sigma$. Shocks to η_t are correlated with stock returns through dm_t . Will amplify risk premia (bad returns have a double bite), but returns will be *positively* correlated with expected returns.
- ▶ Assume investor has CRRA utility with RRA γ over W_T .
- ▶ Can solve using standard methods. E.g., for dynamic programming approach, conjecture $V(W, m, t) = \frac{1}{1-\gamma} W^{1-\gamma} f(m, t)$, and then obtain

$$\phi_t = \frac{m_t - r + v_t \frac{f_m(m_t, t)}{f(m_t, t)}}{\gamma \sigma^2}.$$

Parameter Uncertainty: Portfolio Choice

$$\phi_t = \frac{m_t - r + v_t \frac{f_m(m_t, t)}{f(m_t, t)}}{\gamma \sigma^2}.$$

- ▶ Standard myopic component: $(m_t - r)/(\gamma \sigma^2)$.
- ▶ **New** hedging component to demand: second term, of same sign as f_m .
- ▶ Value function $V(W, m, t) = \frac{1}{1-\gamma} W^{1-\gamma} f(m, t)$ must be increasing in m , so $f_m < 0$ for $\gamma > 1$.
- ▶ Conclude in this case that portfolio demand is lower due to parameter uncertainty: want to hedge against shocks to expected returns (similar to usual ICAPM).
- ▶ Transitory in nature: v_t drifts towards 0.
- ▶ In GE, to ensure $\phi_t = 1$, must have subjective equity premium $m_t - r = \gamma \sigma^2 - v_t \frac{f_m(m, t)}{f(m, t)} \geq \gamma \sigma^2$.
- ▶ See Brennan (1998).

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Imperfectly Observable State Variables: Setup

- ▶ Complete markets, but now consider GE problem for representative investor following Veronesi (2000).
- ▶ Infinite horizon, CRRA utility $E[\int_0^\infty e^{-\rho t} c_t^{1-\gamma} / (1-\gamma) dt]$.
- ▶ Endowment (dividend) process:

$$\frac{d\delta_t}{\delta_t} = \mu_{\delta,t} dt + \sigma_\delta dZ_t.$$

- ▶ Drift $\mu_{\delta,t}$ is unobservable and can take one of several possible values: $\mu_{\delta,t} \in \{\mu_1, \dots, \mu_n\}$.
Observe noisy signal:

$$de_t = \mu_{\delta,t} dt + \sigma_e dZ_{e,t}, \quad dZ_{e,t} dZ_{\delta,t} = 0.$$

- ▶ Actual $\mu_{\delta,t}$ evolves according to Markov chain (Poisson jump process): Constant rate p , so probability pdt of state change over window dt .
- ▶ Assume transitions to state i are independent of current state and have probability given jump of f_i , $i = 1, \dots, n$, so prob. $pf_i dt$ that $\mu_{\delta,t}$ will jump to μ_i over dt .

Imperfectly Observable State Variables: Setup

$$\frac{d\delta_t}{\delta_t} = \mu_{\delta,t} dt + \sigma_{\delta} dZ_t,$$

$$de_t = \mu_{\delta,t} dt + \sigma_e dZ_{e,t}, \quad dZ_{e,t} dZ_{\delta,t} = 0 \quad (\text{signal}),$$

$$\text{Prob}_t(\text{jump to } \mu_i) dt = p f_i dt.$$

- Markov switching filter analogue in this case (Liptser and Shirayev, 1977): Probability of current state being μ_i is $\pi_{i,t}$, which evolves as

$$d\pi_{i,t} = p(f_i - \pi_{i,t}) dt + \pi_{i,t}(\mu_i - m_t)(h_{\delta} d\tilde{Z}_{\delta,t} + h_e d\tilde{Z}_{e,t}),$$

$$m_t = E_t[\mu_{\delta,t}] = \sum_{i=1}^n \pi_{i,t} \mu_i,$$

$$d\tilde{Z}_{\delta,t} = h_{\delta} \left(\frac{d\delta_t}{\delta_t} - m_t dt \right),$$

$$d\tilde{Z}_{e,t} = h_e (de_t - m_t dt),$$

with $h_{\delta} = \sigma_{\delta}^{-1}$, $h_e = \sigma_e^{-1}$.

Imperfectly Observable State Variables: Equilibrium

- Prices from Euler equation:

$$\begin{aligned} S_t &= E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{u'(\delta_s)}{u'(\delta_t)} \delta_s ds \right] \\ &= \sum_{i=1}^n \pi_{i,t} E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{u'(\delta_s)}{u'(\delta_t)} \delta_s ds \middle| \mu_{\delta,t} = \mu_i \right] \\ &= \sum_{i=1}^n \pi_{i,t} C_i, \quad C_i = E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{u'(\delta_s)}{u'(\delta_t)} \delta_s ds \middle| \mu_{\delta,t} = \mu_i \right] \end{aligned}$$

- Itô's Lemma: $stoch\left(\frac{dS_t}{S_t}\right) = (\sigma_\delta + V_t h_\delta) d\tilde{Z}_{\delta,t} + h_e V_t d\tilde{Z}_{e,t}$, where

$$V_t = \frac{\sum_{j=1}^n \pi_{j,t} C_j (\mu_j - m_t)}{\sum_{i=1}^n \pi_{i,t} C_i}.$$

- Using CCAPM equation, get equity premium $\gamma(\sigma_\delta^2 + V_t)$. Can show $V_t < 0$, so that incomplete info *lowers* risk premium. Intuition: Incomplete info of this kind weakens link between returns and consumption growth.

Overreaction to Bad News in Good Times

- ▶ Veronesi (1999) starts from the same point, but specializes to a two-state setting. Equity prices as a function of probability that current state is the high dividend growth state:

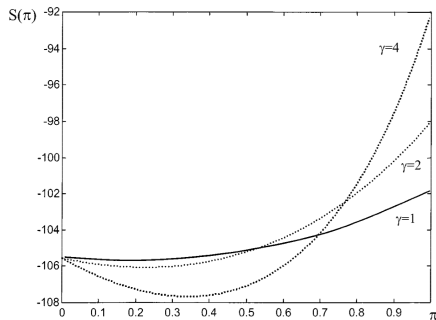


Figure 2
The discount functions $S(\pi)$ for $\gamma = 1, 2, 4$
All functions normalized to have a common starting value.

- ▶ In good times (high π), bad news is doubly bad: it also brings higher uncertainty about future dividend growth (π closer to 0.5). In bad times, good news tempered by higher uncertainty.
- ▶ Generates interesting non-monotonic volatility, and excess volatility w.r.t. dividend news.

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Belief Disagreement

- ▶ Imperfect information is an appealing starting point...
- ▶ ... but maybe not the full story. Lots of disagreement exists in the world. How does this affect prices?
- ▶ Nice recent model: Martin and Papadimitriou (2022).
- ▶ Starting point: Geanakoplos (2010) model, but with risk aversion (rather than portfolio constraints) and many periods (ultimately continuous-time limit).
- ▶ Prices reflect winners' beliefs, which creates more volatility in prices than fundamentals & induces speculation.
- ▶ Will use Ian's slides directly, and then some slides from a discussion of mine.

Sentiment and speculation in a market with heterogeneous beliefs

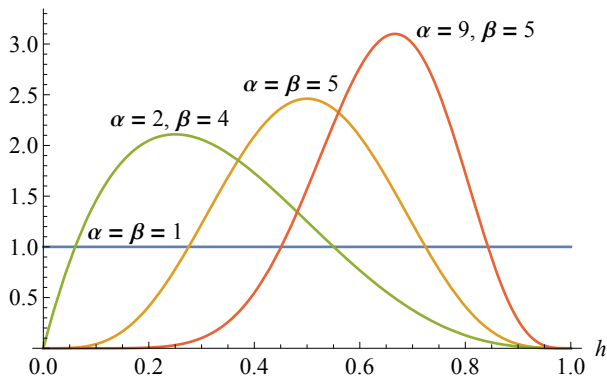
Ian Martin Dimitris Papadimitriou

LSE

King's College London

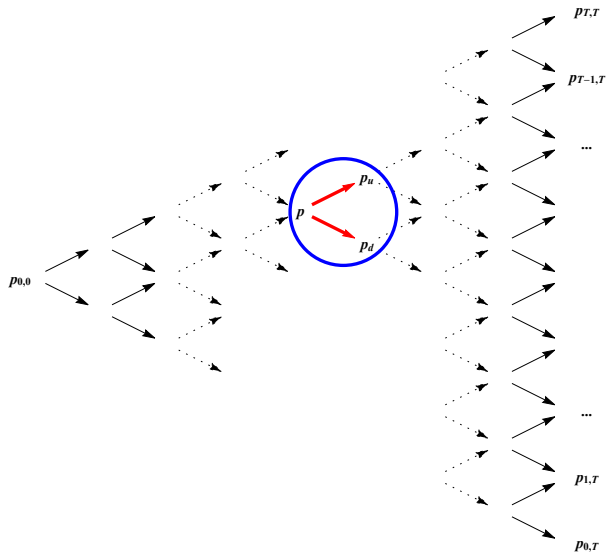
Setup

- All investors are endowed with one unit of a risky asset which evolves on a binomial tree with exogenous terminal payoffs
- Investor $h \in (0, 1)$ thinks the probability of an up-move is h
- Investors have log utility over terminal wealth
- The interest rate is normalized to zero
- No learning (today; see the paper for results with learning)



- Paper handles arbitrary belief distributions
- Today, beta distribution, pdf $f(h) \propto h^{\alpha-1}(1-h)^{\beta-1}$ where $\alpha, \beta > 0$: lets us consider Brownian and Poisson limits

Log investors are myopic



Equilibrium (1): individual optimization

- Solve backwards: the price of the risky asset is p_d or p_u next period
- Agent h has wealth w_h and holds x_h units of the asset (price p)
- So portfolio problem is

$$\max_{x_h} h \log \underbrace{[w_h - x_h p + x_h p_u]}_{\text{wealth in up state}} + (1 - h) \log \underbrace{[w_h - x_h p + x_h p_d]}_{\text{wealth in down state}}$$

- First order condition:

$$x_h = w_h \left(\frac{h}{p - p_d} - \frac{1 - h}{p_u - p} \right)$$

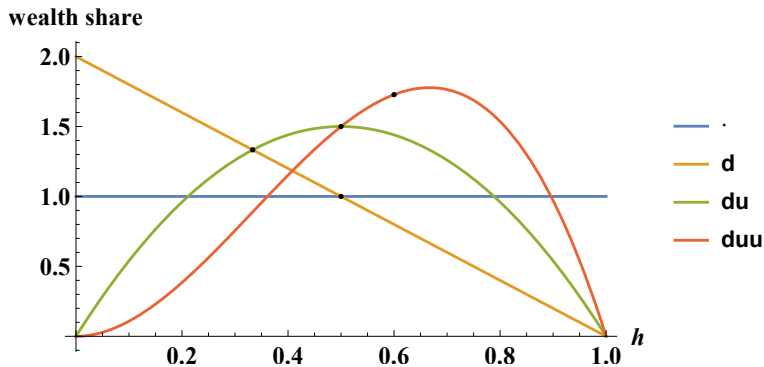
- Helpful to rewrite the FOC in terms of the risk-neutral probability of an up-move, h^* , which is defined via $p = h^*p_u + (1 - h^*)p_d$
- The realized return on wealth, for agent h , is then

$$\frac{h}{h^*} \quad \text{in the up state;} \quad \frac{1-h}{1-h^*} \quad \text{in the down state}$$

- So after m up and n down steps, agent h 's wealth is $\lambda_{\text{path}} h^m (1-h)^n$
- To pin down λ_{path} , note that aggregate wealth equals p , so

$$w_h = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} p h^m (1-h)^n$$

- The richest agent is $h = m/(m+n)$, who looks right in hindsight



- Figure assumes uniform distribution of beliefs, i.e., $\alpha = \beta = 1$
- Less disagreement \implies smaller shifts in wealth distribution

Equilibrium (2): market clearing

- From the FOC,

$$x_h = \underbrace{\frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)}}_{w_h} p h^m (1 - h)^n \left(\frac{h}{p - p_d} - \frac{1 - h}{p_u - p} \right)$$

- The equilibrium price ensures that, in aggregate, agents hold one unit of the asset:

$$p = \frac{(m + n + \alpha + \beta)p_u p_d}{(m + \alpha)p_d + (n + \beta)p_u}$$

A general pricing formula

Result

If the risky asset has terminal payoffs $p_{m,T}$ then its initial price is

$$p_0 = \frac{1}{\sum_{m=0}^T \frac{c_m}{p_{m,T}}}$$

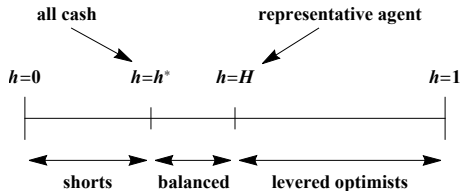
where

$$c_m = \binom{T}{m} \frac{B(\alpha + m, \beta + T - m)}{B(\alpha, \beta)}$$

Result (The effect of sentiment)

The price p_0 falls as disagreement increases if $\frac{1}{p_{m,T}}$ is convex in m (and rises if $\frac{1}{p_{m,T}}$ is concave)

Two special investors



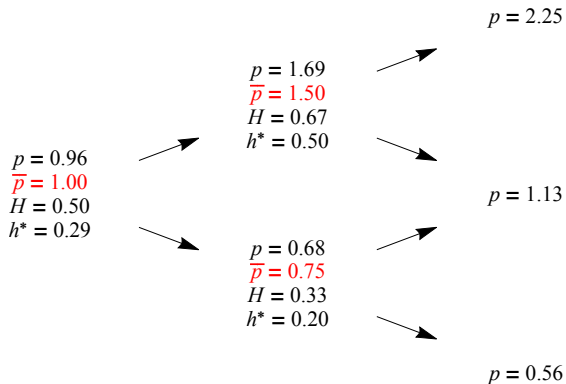
- In equilibrium,

$$\text{risky share of agent } h = \frac{h - h^*}{H - h^*} \quad \text{where} \quad H = \frac{m + \alpha}{m + n + \alpha + \beta}$$

- $h = H$ is the rep agent—"Mr. Market"
- $h = h^*$ is out of the market—a bond investor who's fully in cash

An example

$$\alpha = \beta = 1$$



p : price. \bar{p} : price in homogeneous economy. H : rep agent
 h^* : cash investor (cutoff between longs and shorts)

- Agents disagree on the risk premium

$$\text{agent } h\text{'s perceived risk premium} = \frac{(h - h^*)(H - h^*)}{h^*(1 - h^*)}$$

- But they agree on objectively measurable quantities, such as

$$\text{risk-neutral variance} = \frac{(H - h^*)^2}{h^*(1 - h^*)}$$

or

$$\text{VIX}^2 = 2 \left[h^* \log \frac{h^*}{H} + (1 - h^*) \log \frac{1 - h^*}{1 - H} \right]$$

- Notice that

$$\text{risky share of agent } h = \frac{h - h^*}{H - h^*} = \frac{\text{agent } h\text{'s risk premium}}{\text{risk-neutral variance}}$$

- In particular, the risk premium perceived by Mr. Market equals risk-neutral variance

Example 1: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
 - ▶ $h = 0.50$ thinks default prob is less than 10^{-15}
 - ▶ $h = 0.25$ thinks default prob is less than 10^{-6}
 - ▶ $h = 0.10$ thinks default prob is less than 0.006
 - ▶ $h = 0.05$ thinks default prob is less than 8%
 - ▶ $h = 0.01$ thinks default prob is more than 60%
- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at?

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- Who will *stay* short?

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- Who would go short, at this price? everyone below $h = 0.48!$
- Who will stay short? marginal agent h^* in period 0, 1, 2, ... is $h = 0.48, 0.31, 0.22, \dots$

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- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at \$95.63
- Who would go short, at this price? everyone below $h = 0.48!$
- Who will stay short? marginal agent h^* in period 0, 1, 2, ... is $h = 0.48, 0.31, 0.22, \dots$; only $h < 0.006$ stay short to the bitter end

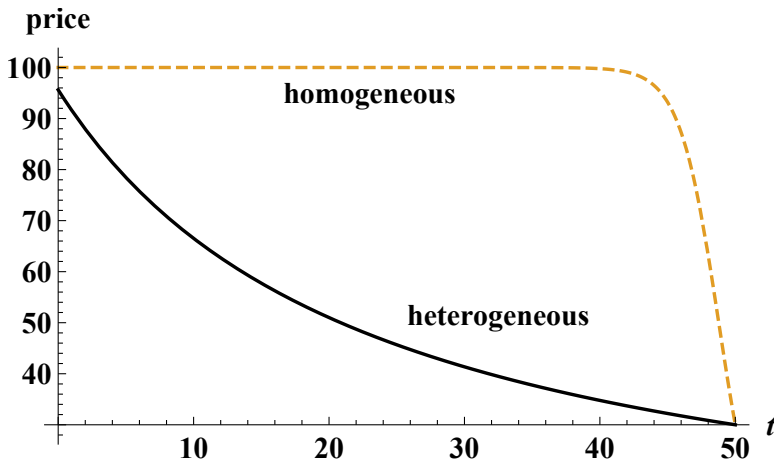


Figure: The risky bond's price over time following consistently bad news

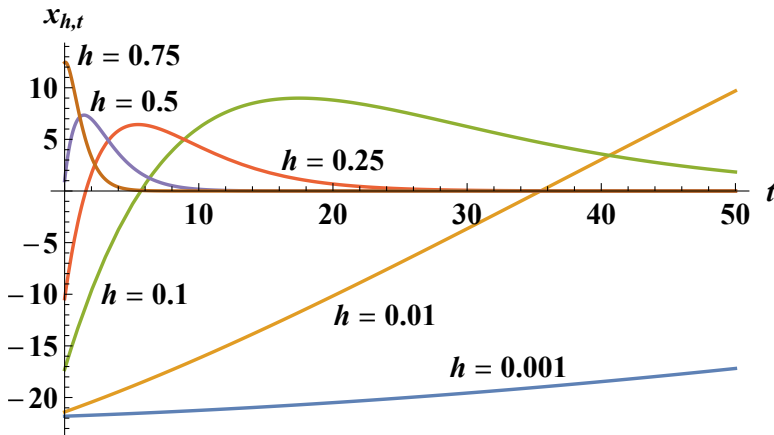


Figure: The number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time

- Price is low at time zero because all investors—even “reasonable” ones—worry about the short-term effect of bad news on sentiment
- The risk-neutral probability of default, δ^* , is 6.25%

$$\delta^* = \frac{1}{1 + \varepsilon T} = O(1/T)$$

- In the homogeneous economy, it is less than 10^{-14}

$$\delta^* = \frac{1}{1 + \varepsilon (2^T - 1)} = O(2^{-T})$$

- Polynomial / exponential dichotomy holds for any finite α, β ; and if “recovery value” is greater than 100 (bubbly asset)
- Sentiment makes long-dated extreme securities far more valuable

Speculative strategies vs. fundamental views

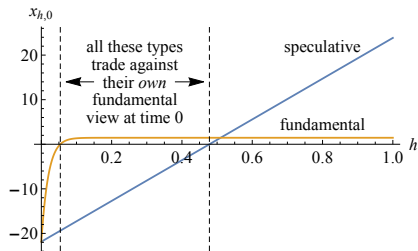


Figure: Positions of different investors at time 0 under dynamic (“speculative”) and static (“fundamental”) trade

- Investor $h = 0.25$ thinks there's less than a 10^{-6} chance of default, so risky bond is almost sure to deliver an excess return $\sim 5\%$
- Nonetheless, goes **short** initially to speculate on sentiment

Example 2: Risky bond vs. bubbly asset

Left-skewed (risky bond)

- Sentiment drives price **down**
- Price drop occurs **early**
- Volatility **declines** over time
- Median investor **increasingly bullish**

Right-skewed (bubbly asset)

- Sentiment drives price **up**
- Bubble emerges **late**
- Volatility **rises** over time
- Median investor **bullish, then bearish, then bullish**

- Risk drives the price **toward** the worst-case scenario for left-skewed asset, and **away** from the best-case scenario for right-skewed asset
- Result: it's all over more quickly for left-skewed asset. High vol and risk premia late in the game for right-skewed asset

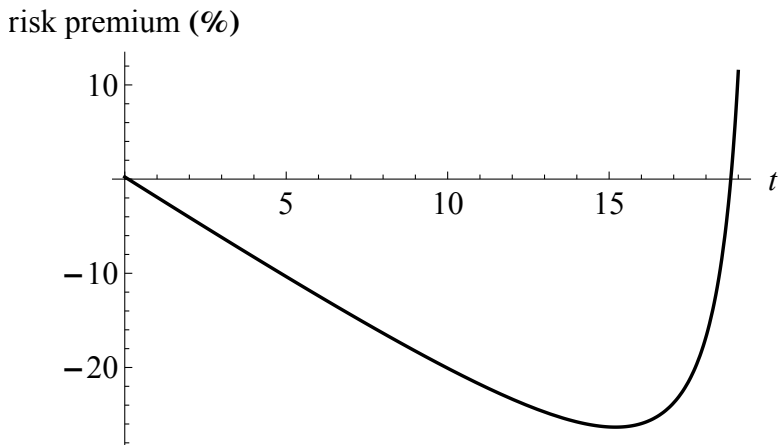


Figure: Median investor's expected excess return on the bubbly asset

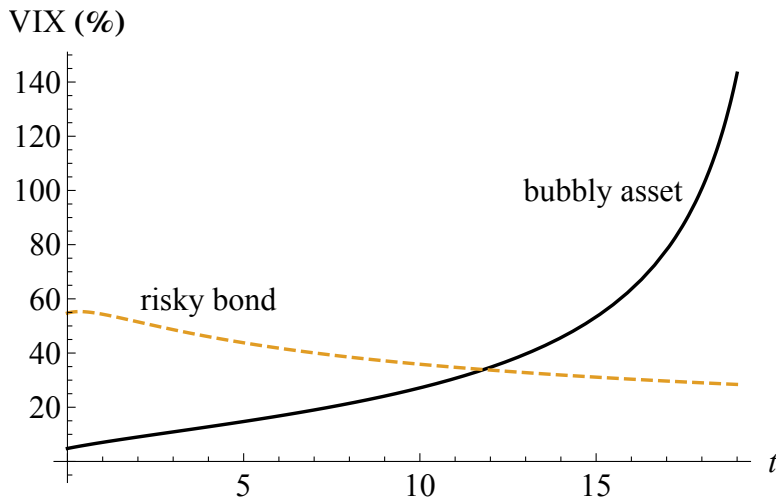


Figure: VIX over time following consistently good/bad news

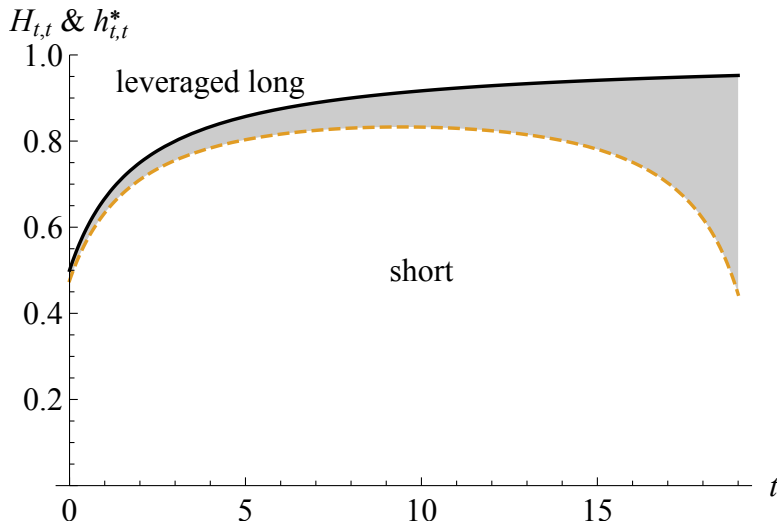


Figure: Identities of the rep investor and cash investor over time

Example 3: A diffusion limit

- Slice the period from 0 to T into $2N$ short periods
- Cox–Ross–Rubinstein terminal payoffs, $p_{m,T} = e^{2\sigma\sqrt{\frac{T}{2N}}(m-N)}$
- Tune down per-period disagreement by parametrizing $\alpha = \beta = \theta N$
- Low θ : lots of disagreement. $\theta \rightarrow \infty$: homogeneous economy
- Convenient to index agents by their z -score, the number of s.d. by which they are more/less optimistic than the mean
- As $N \rightarrow \infty$, everyone perceives the risky return as lognormal
- This is a world in which people agree on second moments (volatility) but disagree on first moments (the risk premium)

Result (Subjective expectations)

The (annualized) expected return of the asset from 0 to t from the perspective of a trader z is:

$$\frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} = \frac{\theta + 1}{\theta + \frac{t}{T}} \left[\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta + 1}{\theta} \frac{\theta + \frac{t}{2T}}{\theta + \frac{t}{T}} \sigma^2 \right]$$

In particular, the cross-sectional average expected return is

$$\tilde{\mathbb{E}} \frac{1}{t} \log \mathbb{E}^{(z)} R_{0 \rightarrow t} = \frac{(\theta + 1)^2 \left(\theta + \frac{t}{2T}\right)}{\theta \left(\theta + \frac{t}{T}\right)^2} \sigma^2$$

Disagreement is the cross-sectional standard deviation of expected returns:

$$\text{disagreement} = \frac{\theta + 1}{\theta + \frac{t}{T}} \frac{\sigma}{\sqrt{\theta T}}$$

Result (Option pricing and the volatility term structure)

The time 0 price of a call option with maturity t and strike price K obeys the Black–Scholes formula with implied volatility

$$\tilde{\sigma}_t = \frac{\theta + 1}{\sqrt{\theta(\theta + \frac{t}{T})}} \sigma$$

In particular, short-dated options have $\tilde{\sigma}_0 = \frac{\theta+1}{\theta}\sigma$ and long-dated options have $\tilde{\sigma}_T = \sqrt{\frac{\theta+1}{\theta}}\sigma$. As all agents agree on true volatility

$$\sigma_t^{(z)} = \left(\frac{\theta + 1}{\theta + \frac{t}{T}} \right) \sigma,$$

there is a variance risk premium $\frac{1}{T} (\text{var}^ \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T}) = \frac{\sigma^2}{\theta}$*

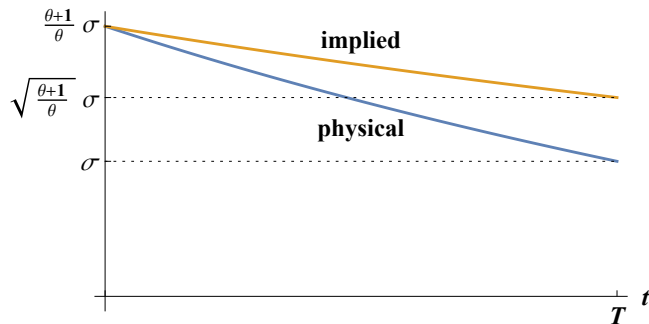


Figure: The term structures of implied and physical volatility

- Variance risk premium $\frac{1}{T} (\text{var}^* \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T}) = \frac{\sigma^2}{\theta}$

An illustrative calibration

	Data	Model
1mo implied vol	18.6%	18.6%
1yr implied vol	18.1%	18.2%
2yr implied vol	17.9%	17.7%
1yr cross-sectional mean risk premium	3.8%	3.2%
1yr disagreement	4.8%	4.2%
10yr cross-sectional mean risk premium	3.6%	1.8%
10yr disagreement	2.9%	2.8%

- $T = 10$, $\sigma = 12\%$, $\theta = 1.8$
- Despite being highly stylized, the model generates predictions of broadly the right order of magnitude across multiple dimensions

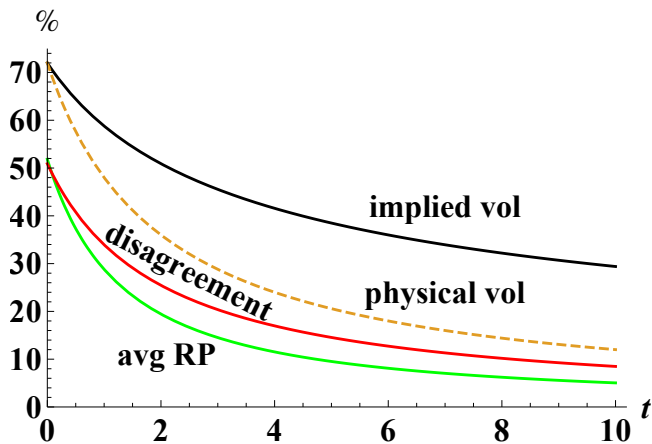


Figure: Volatility term structures in a “crisis” calibration with $\theta = 0.2$

Why is there a variance risk premium?

- We introduce an identity

$$\text{var}^* X - \text{var} X = R_f \text{cov} \left[M, (X - \kappa)^2 \right]$$

for any tradable X , where $\kappa = (\mathbb{E}X + \mathbb{E}^* X)/2$ is a constant

- This is a **general result**, relying only on absence of arbitrage
- In the mind of our median investor, it specializes to

$$\text{var}^* \log R_{0 \rightarrow T} - \text{var} \log R_{0 \rightarrow T} = \underbrace{\text{cov}^{(z)} \left[M_{0 \rightarrow T}^{(z)}, (\log R_{0 \rightarrow T})^2 \right]}_{\text{zero in Black-Scholes, positive here—but why?}}$$

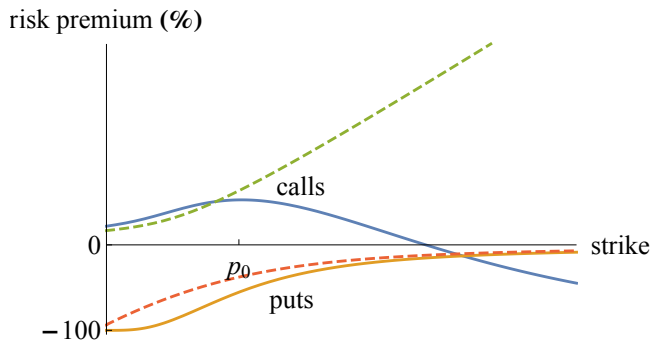


Figure: Expected excess returns on options of different strikes, as perceived by the rep agent. Solid: heterogeneous beliefs. Dashed: homogeneous

- Median agent thinks OTM options are overvalued due to extremists
- Perceives *negative* expected excess returns on deep OTM calls

Example 4: A Poisson limit

- Bad news arrives according to a Poisson process
- If q arrivals occur, terminal payoff is e^{-qJ} (for some constant J)
- Agents disagree on the jump arrival rate ω and hence on all moments of returns
- Optimists perceive low arrival rates and sell insurance to pessimists; like derivative traders inside financial institutions, they do well in quiet times but experience losses at times of turmoil
- As before, we have a representative agent ($\omega_{\text{rep},t}$) and an agent who is out of the market (ω_t^* ; and ω_t^* = the CDS rate)
- Both more pessimistic ($\omega_{\text{rep},t}$ and ω_t^* get larger) following jumps
- All investors think **arbitrarily** high Sharpe ratios are attainable

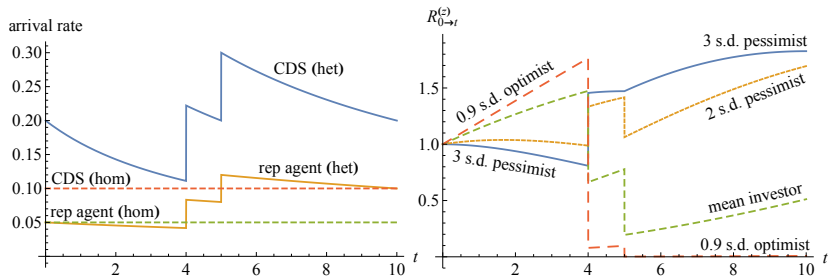


Figure: Left: $\omega_{\text{rep},t}$ and ω_t^* on a sample path with jumps at times $t = 4$ and 5
 Right: The cumulative return of four agents along the same sample path

- Even though individuals have stable beliefs, the CDS rate and rep agent's perceived arrival rate spike **after** a jump

Speculation is a mixed blessing

- All investors think speculation is in their own interest
- But all investors also think that speculation is socially costly
- On the other hand, if speculation (i.e., dynamic trade) is closed down entirely, the market can collapse
- To see what can go wrong, consider the Brownian limit. . .

Speculation is a mixed blessing

- Given any positive time 0 price, return to maturity is lognormal
- So a **static** short or levered position risks bankruptcy
- Therefore, agent z 's risky share $\in [0, 1]$
- To clear the market, the average risky share must be 1. So *all* agents must choose risky share equal to 1
- But this is impossible! At any fixed positive price, some investors will not wish to invest fully in the risky asset
- Hence static equilibrium does not exist
- Although speculation is socially costly, the ability to trade dynamically means investors can reduce their position sizes to avoid bankruptcy if the market starts to move against them

Summary

- Sentiment creates volatility, ambiguous impact on risk premia
- Extreme scenarios are important for pricing
- Asymmetric effects on right- and left-skewed assets
- Moderate investors are contrarian, “short vol”, liquidity suppliers
- Mean-variance-efficient returns are very short deep-OTM options; they do not interest our investors despite their high Sharpe ratios
- CDS rates spike after jumps, even though all investors perceive constant arrival rates
- Everyone thinks that speculation is socially harmful, but good news for themselves and for people with similar beliefs

Sentiment and Speculation in a Market with Heterogeneous Beliefs

IAN MARTIN
LSE

DIMITRIS PAPADIMITRIOU
LSE

Discussion:

EBEN LAZARUS
MIT Sloan

NBER SI Asset Pricing
July 2019

Background

Two (overlapping) categories of literature on **belief disagreement**:

1. Heterogeneity + short-sale constraints \implies overvaluation

[Miller (1977), Harrison & Kreps (1978), Scheinkman & Xiong (2003), ...]

- ▶ Useful for explaining speculative bubbles
- ▶ *Not* useful — harmful, in fact — for generating unconditional aggregate equity premium

2. Heterogeneity + borrowing & dynamic trading \implies excess trading & volatility

[Shiller (1984), DeLong, Shleifer, Summers, & Waldmann (1990), David (2008), Banerjee & Kremer (2010), Geanakoplos (2010), Barberis, Greenwood, Jin, & Shleifer (2015), Atmaz & Basak (2018), ...]

- ▶ Rich literature; many different settings & conclusions
- ▶ Papers are either stylized (built to match only a few data features) or technically challenging

This paper lives in the second category. Do we need another paper in that list?

- ▶ Yes! Paper provides useful insights: matches features of aggregate data with elegance & simplicity
- ▶ Seems to me a very useful minimal dynamic model of heterogeneity with complete markets

Outline

1. Summary: Setting and Results
2. Alternative Interpretations
3. Questions

Review: Basic Results

Optimality + market clearing (with some neat algebra using the payoff approach) give:

1. Wealth distribution: Fraction of aggregate wealth p_t held by type- h agents, $\frac{\text{wealth}_{h,t} f(h)}{p_t}$, follows Beta($\alpha + m, \beta + t - m$), where m is # of up moves from 0 to t
 - ▶ Why? Because the beta distribution is the conjugate prior of the binomial distribution, and have assumed beta “prior” distribution of agents and binomial evolution of tree
 - ▶ So beta distribution is the “right” choice for initial wealth distribution
 - ▶ Delivers very clear, closed-form generalization of logic of Geanakoplos (2010): wealth accrues to investors who are correct in hindsight
2. Pricing: At any date t , after m up moves, the risky asset's price $p_{m,t}$ is

$$p_{m,t} = \frac{1}{\sum_{m'=0}^{T-t} \text{Prob}_{\text{RepAgent},t}[(\text{up moves from } t \text{ to } T) = m'] \times p_{m+m',T}^{-1}}$$

- ▶ This is “just” the harmonic-mean payoff perceived by the (wealth-weighted) rep. agent
- ▶ Why harmonic mean? Because of log utility
- ▶ What beliefs does this representative agent hold? More interpretation in a few slides, but note that bad news is amplified by pessimists becoming wealthier (& vice versa), *and* this is priced

Additional Results and Implications

- (i) For very general payoffs as function of # of up moves, $p_{m,T}$, the risky asset's expected return is increasing in belief heterogeneity \Rightarrow **Equity premium** ✓
- (ii) In good times (as the wealth-weighted avg. belief increases), *all* individual investors believe the market's Sharpe ratio is **lower**, but it can be shown that $\frac{dSR_{RepAgent,t}}{dRepAgentBelief_t} > 0$ (should include this!)
 - ▶ So while all individual investors underreact to new information (by design), the market **overreacts** to good news in the sense that it perceives a higher Sharpe ratio in good times \Rightarrow **survey evidence** [Bordalo, Gennaioli, Ma, Shleifer (2018)] ✓
& **excess volatility of prices and Sharpe ratios** [Shiller (1981), ..., Lazarus (2018)] ✓
- (iii) Term structure of expected returns (as perceived by all agents) is downward sloping, with greater downward slope in bad times \Rightarrow **term structure and cyclicity of risk premia** ✓
 - ▶ Same for term structures of implied and physical volatility
- (iv) And all of this with a constant risk-free rate

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1. Summary: Setting and Results
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Interpretation of Results: Decompositions

- ▶ Multiple new modeling choices here relative to previous benchmarks; how much does each contribute to results? Will focus just on prices
- ▶ Across all models under consideration:
 - ▶ Normalize $\mathbb{E}_0[p_{m,T}] = e$ (where expectation is w.r.t. representative agent's beliefs)
 - ▶ Assume agents are symmetrically distributed around up-move belief $h = 1/2$ (all equal to $h = 1/2$ in homogeneous-agent case, and $\alpha = \beta = \theta N$ in heterogeneous case)
 - ▶ Work in continuous-time limit
- ▶ **My benchmark model:** Homogeneous risk-neutral economy, $p_0 = \mathbb{E}_0[p_{m,T}] = e$
- ▶ **Decomposition 1:**

$$\begin{aligned}
 \log(p_{0,\text{heterogeneous}}/p_{0,\text{benchmark}}) &= \log(p_{0,\text{heterogeneous}}/e) = \log(p_{0,\text{heterogeneous}}) \\
 &= \underbrace{\log(p_{0,\text{heterogeneous}}/p_{0,\text{homogeneous,risk-averse}})}_{\text{effect of heterogeneity}} + \underbrace{\log(p_{0,\text{homogeneous,risk-averse}})}_{\text{effect of risk aversion}} \\
 &= \underbrace{-\frac{1}{2\theta}}_{\text{heterogeneity}} + \underbrace{-\frac{1}{2}}_{\text{risk aversion}} \stackrel{\text{main calibration}}{=} -0.28 - 0.5
 \end{aligned}$$

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- ▶ **Decomposition 2:**

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\implies rep. agent's belief in heterogeneous economy is *equal* to belief held by single agent in a learning economy (“wisdom of the crowd”)

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\Rightarrow further, we learn that in isolation, uncertainty and disagreement work in exactly the same direction here