A New Test of Excess Movement in Asset Prices

NED AUGENBLICK
UC Berkeley Haas

EBEN LAZARUS
MIT Sloan

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Background

Asset prices are volatile. Is this volatility **informative for expectations formation** in GE?

- For example: Too volatile to be consistent with rationality?
- Unanswerable without further structure...e.g., classic volatility bounds [Shiller (1981)]:

$$P_t + \text{error} = \text{ex-post fundamental value}$$

$$\implies \quad \text{Var}(P_t) < \text{Var}(\text{ex-post fundamental value}) \qquad \text{[Theory]}$$

$$\text{Var}(P_t) > \text{Var}\left(\sum_{j=1}^{\infty} \frac{D_{t+j}}{\underbrace{R^j}_{\text{constant discount rates}}}\right) \qquad \text{[Data]}$$

- Response [Fama (1991)]:

  *"Volatility tests are a useful way to show that expected returns vary, [but] give **no** help on the central issue of whether the variation in expected returns is rational."*

- Can further statements be made with less structure?

Background

Asset prices are volatile. Is this volatility **informative for expectations formation**?

- Should we care?
- Rich and growing literature on expectations formation in macro & finance making use of survey data
- But questions remain:
  1. Mapping from survey responses to high-stakes behavior [Cochrane (2017), Manski (2018)]
  2. GE outcomes [Angeletos, Huo, Sastry (2021)]
- Will make some progress on these questions after presenting & estimating our bounds
- Bounds also give new info on interaction of beliefs & risk aversion for broad class of models

What We Do

- Like Shiller, focus on expectations over future equity index value. . .
- . . .but consider behavior of *options* written on future index value, rather than underlying index itself
- Apparently minor change in focus gives significant theoretical traction:
    1. Multiple options on same index with same expiration $\Longrightarrow$ comparing *relative* prices allows us to discard price variation arising from discounting & common unobservable shocks
    2. Dynamics can be restricted without knowledge of true fundamental value
- Focus on equity index for interpretation & empirical implementation, but theoretical results apply generally to rational valuation processes for state-contingent payoffs

Results

1. **Theory:** In general framework, derive bound under RE:

$$\underset{\text{[risk-neutral beliefs]}}{\text{Variation in option-implied beliefs}} \leqslant f(\underset{\text{[SDF slope]}}{\text{risk aversion}})$$

- Main joint assumption: $\frac{\mathbb{E}_t[M_T \mid \text{return state } a]}{\mathbb{E}_t[M_T \mid \text{return state } b]}$ constant over t within an option contract

  [met in range of standard frameworks, and generates informative joint null]
- Logic of bound: Imagine observing beliefs over binary outcome at date T:

$$\pi_0 = 0.1 \rightarrow \pi_1 = 0.9 \rightarrow \pi_2 = 0.1 \rightarrow \pi_3 = 0.9 \rightarrow \dots$$

- Possible that this was generated by extreme signals...
- ...but if we observe repeatedly, likely a violation of Bayes' rule w.r.t. true DGP
- Show that this logic can be extended to risk-neutral beliefs given joint assumption

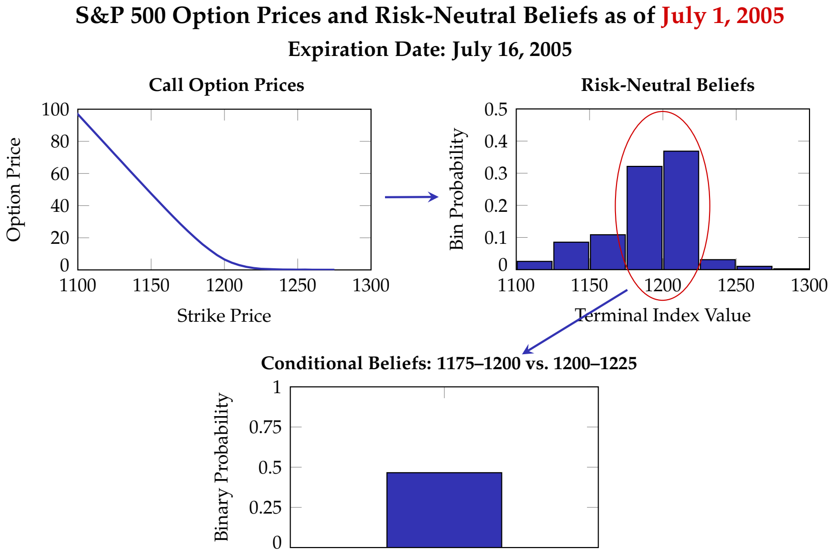
Results

1. **Theory:** In general framework, derive bound under RE:

$$\underset{\text{[risk-neutral beliefs]}}{\text{Variation in option-implied beliefs}} \leqslant \underset{\text{[SDF slope]}}{f(\text{risk aversion})}$$

2. **Data:**
   - S&P index options
   - Volatile risk-neutral beliefs $\Longrightarrow$ very high required risk aversion & frequent bounds violations
   - Excess movement in RN beliefs comoves strongly with excess movement in individual SPF forecasts of output growth & inflation

Preview: Empirical Variation



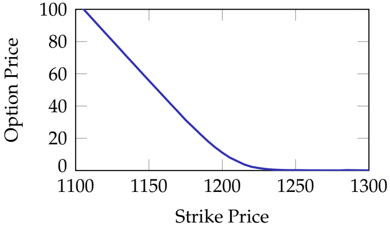
Intuition

Preview: Empirical Variation

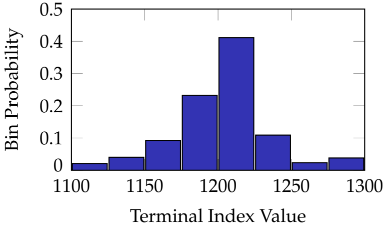
S&P 500 Option Prices and Risk-Neutral Beliefs as of July 5, 2005

Expiration Date: July 16, 2005

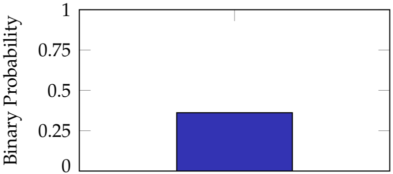
Call Option Prices



Risk-Neutral Beliefs



Conditional Beliefs: 1175–1200 vs. 1200–1225



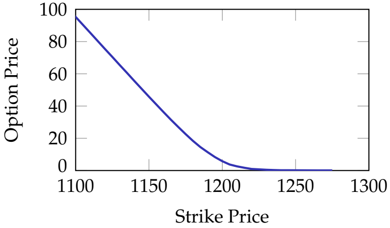
Intuition

Preview: Empirical Variation

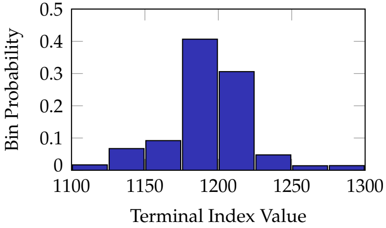
S&P 500 Option Prices and Risk-Neutral Beliefs as of July 6, 2005

Expiration Date: July 16, 2005

Call Option Prices



Risk-Neutral Beliefs



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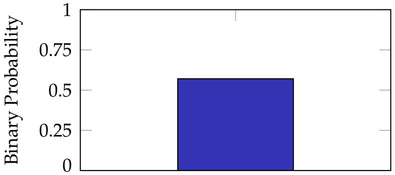
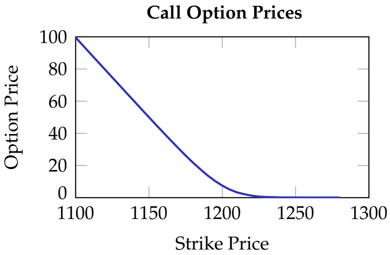


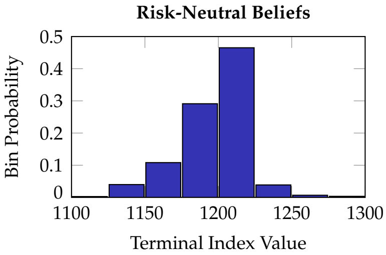
Intuition

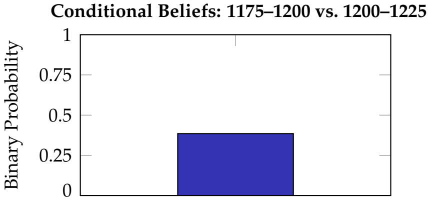
Preview: Empirical Variation

S&P 500 Option Prices and Risk-Neutral Beliefs as of July 7, 2005

Expiration Date: July 16, 2005







Intuition

Outline

1. Introduction

2. Theory
   Two-State Setting
   General AP Setting

3. Evidence from Index Options

4. Excess Movement and Aggregate Statistics

5. Discussion and Conclusions

Two-State Setting

How can we characterize AD state prices under RE? Will start simple and build piece by piece.

Setting (*assumptions to be dropped*):

- Discrete time, $t = 0, 1, \ldots, T$
- Single agent
- Two states: $\theta \in \{0, 1\}$, and realization determines terminal consumption ($C_T : \theta \to \mathbb{R}_+$)
- Signals $s_t \in \mathcal{S}$ drawn from discrete distribution $DGP(s_t \mid \theta, H_{t-1})$, where H_{t-1} is signal history
- Write $\mathbb{P}(H_T)$ for prob. of observing H_T induced by DGP ($\mathbb{E}[\cdot] \equiv \mathbb{E}^{\mathbb{P}}[\cdot]$)
- Beliefs: $\pi_t(H_t) \equiv$ subj. prob. for state $\theta = 1$ (vs. state 0)

Assumption 1 (*RE*)

Beliefs satisfy $\pi_t(H_t) = \mathbb{E}[\theta \mid H_t]$ for all H_t.

- Implies $\pi_t = \mathbb{E}[\pi_{t+1} \mid \pi_t]$ (sufficient for main results)

Two-State Setting: Directly Observed Beliefs

How can we characterize AD state prices under RE? Will start simple and build piece by piece.

Setting (*assumptions to be dropped*):

- Observable: Agent's valuation $q_t(\theta)$ of Arrow-Debreu security for $\theta \in \{0, 1\}$
  - Payoff: $\mathbb{1}\{\theta\}$
- In general, cannot directly observe DGP or physical beliefs
- First case: Risk neutrality, no discounting $\implies$ valuations reveal beliefs:

$$q_t(1) = \pi_t, \quad q_t(0) = 1 - \pi_t$$

Two-State Setting: Directly Observed Beliefs

For belief stream π, keep track of:

1. **Belief movement:** $$\mathsf{m}(\boldsymbol{\pi}) \equiv \sum_{t=0}^{T-1} (\pi_{t+1} - \pi_t)^2$$
   - "Volatility" $\Longleftrightarrow$ sum of squared changes in beliefs
2. **Initial uncertainty:** $$\mathsf{u}_0(\boldsymbol{\pi}) \equiv (1 - \pi_0)\pi_0$$
   - "Uncertainty" $\Longleftrightarrow$ variance of Bernoulli RV $\mathbb{1}\{\theta = 1\}$, maximized at 0.5
   - $\mathsf{u}_T = 0$ given $\pi_T \in \{0, 1\}$, so **uncertainty resolution** is $\mathsf{r}(\boldsymbol{\pi}) \equiv \mathsf{u}_0 - \mathsf{u}_T = \mathsf{u}_0$
3. **Excess movement:** $$X(\boldsymbol{\pi}) \equiv \mathsf{m}(\boldsymbol{\pi}) - \mathsf{u}_0(\boldsymbol{\pi})$$

Two-State Setting: Directly Observed Beliefs

For belief stream π, keep track of:

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2. **Initial uncertainty:** $$\mathsf{u}_0(\boldsymbol{\pi}) \equiv (1 - \pi_0)\pi_0$$
3. **Excess movement:** $$X(\boldsymbol{\pi}) \equiv \mathsf{m}(\boldsymbol{\pi}) - \mathsf{u}_0(\boldsymbol{\pi})$$

Lemma 1 (*Augenblick & Rabin, 2021*)

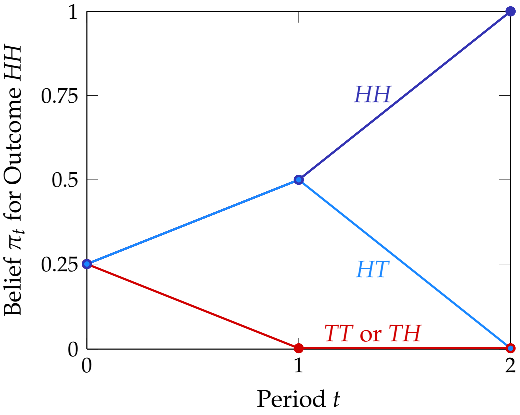
Under RE, for any DGP,
$$\mathbb{E}[X] = 0$$

Derivation

- Formalizes "correct" amount of subjective belief movement
- Derivation uses only martingale property of beliefs
- **Intuition:** Changing beliefs $\iff$ must be learning something (on average)
- Violations can arise from too large (or small) belief revisions

Two-State Example: Directly Observed Beliefs

$$T = 2\text{, sequential fair coin flips at } t = 1 \text{ and } t = 2\text{, } C_T = \begin{cases} C_{\text{low}} & \text{if } HH \quad (\theta = 1) \\ C_{\text{high}} & \text{else} \quad (\theta = 0) \end{cases}$$



Objects: $\mathsf{m} \equiv \sum_{t=0}^{1} (\pi_{t+1} - \pi_t)^2, \; \mathsf{u}_0 \equiv (1 - \pi_0)\pi_0$

Lemma: $\mathbb{E}[X] = 0 \iff \mathbb{E}[\mathsf{m}] = \mathsf{u}_0$

Path	**Movement (m)**	**Frequency ($\mathbb{P}$)**
HH	$(1/2 - 1/4)^2 + (1 - 1/2)^2 = 5/16$	$1/2 \times 1/2 = 1/4$
HT	$5/16$	$1/4$
$T*$	$1/16$	$1/2$

$$\Longrightarrow \quad \mathbb{E}[\mathsf{m}] = 3/16$$

$$= 3/4 \times 1/4 = \mathsf{u}_0 \quad \checkmark$$

Two-State Setting with Risk Aversion

Assume now:

- Utility: $\mathbb{E}_0 \sum_{t=0}^{T} \beta^t \, U(C_t), \;\; U'' < 0$
- State $\theta \in \{0, 1\}$ again determines terminal consumption $C_{T,\theta}$
  - Normalize $C_{T,1} \leqslant C_{T,0} \implies \theta = 1$ is "bad" state
  - No restrictions on intermediate consumption $\{C_t\}$

Two-State Setting with Risk Aversion

Assume now:

- Utility: $\mathbb{E}_0 \sum_{t=0}^{T} \beta^t \, U(C_t), \;\; U'' < 0$
- State $\theta \in \{0, 1\}$ again determines terminal consumption $C_{T,\theta}$, with $C_{T,1} \leqslant C_{T,0}$
- State prices:

$$q_t(1) = \frac{\beta^{T-t} U'(C_{T,1})}{U'(C_t)} \, \pi_t, \quad q_t(0) = \frac{\beta^{T-t} U'(C_{T,0})}{U'(C_t)} (1 - \pi_t)$$

$\Longrightarrow$ subjective beliefs no longer observable

- As $q_t(1)$ and $q_t(0)$ are similarly distorted by $\beta^{T-t}/U'(C_t)$, consider **risk-neutral (RN)** belief:

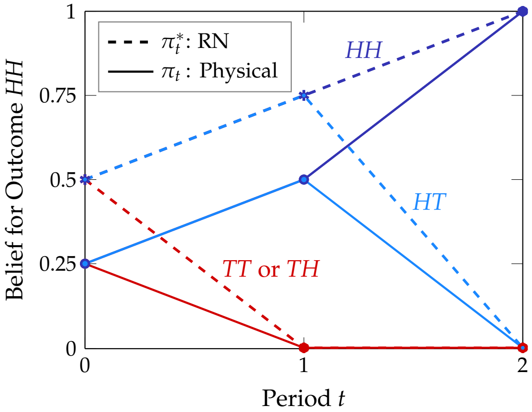
$$\pi_t^* \equiv \frac{q_t(1)}{q_t(0) + q_t(1)} = \frac{U'(C_{T,1})}{\mathbb{E}_t[U'(C_T)]} \, \pi_t = \frac{\phi \pi_t}{1 + (\phi - 1)\pi_t} \geqslant \pi_t$$

$$\text{where} \;\; \phi \equiv \frac{U'(C_{T,1})}{U'(C_{T,0})} = \frac{\mathrm{SDF}_T(1)}{\mathrm{SDF}_T(0)}$$

  - Challenge: π_t^* need not follow a $\mathbb{P}$-martingale under RE $\Longrightarrow$ can have $\mathbb{E}[X^*] > 0$

Two-State Example: Risk-Neutral Beliefs

$T = 2$, sequential coin flips, $\theta = 1$ if HH, and $\phi = 3 \iff U'(C_{T,1}) = 3 \times U'(C_{T,0})$



Observe: $\mathsf{m}^* \equiv \sum_{t=0}^{1} (\pi_{t+1}^* - \pi_t^*)^2, \ \mathsf{u}_0^* \equiv (1 - \pi_0^*)\pi_0^*$

$$\mathbb{E}[X^*] \stackrel{?}{=} 0 \iff \mathbb{E}[\mathsf{m}^*] \stackrel{?}{=} \mathsf{u}_0^*$$

Path	RN Movement (m^*)	Frequency ($\mathbb{P}$)
HH	$\cancel{5/16}$ $1/8$	$1/4$
HT	$\cancel{5/16}$ $5/8$	$1/4$
$T*$	$\cancel{1/16}$ $1/4$	$1/2$

$\implies \quad \mathbb{E}[\mathsf{m}^*] = 5/16 > 1/2 \times 1/2 = \mathsf{u}_0^*$ ✘

$\mathbb{E}[X^*] > 0$

$\mathbb{E}[X^*] \leqslant$?

Two-State Setting: Results

How much RN excess movement X^* can there be?

- The possibility that $\mathbb{E}[X^*] \geqslant 0$ seems to suggest anything goes...but not the end of the story:
  1. RN beliefs not arbitrarily distorted relative to physical beliefs: $\pi_t^* = \frac{\phi\pi_t}{1+(\phi-1)\pi_t}$
  2. For *any* ϕ, RN beliefs bounded by definition: $\pi_t^* \in [0,1]$
  3. $\mathbb{E}^*[X^*] = 0$
- Taken together and maximizing $\mathbb{E}[X^*]$ over all possible DGPs, obtain bound:

Result 1

Under RE, for any DGP,

$$\mathbb{E}[X^*] \leqslant \pi_0^* \underbrace{(\pi_0^* - \pi_0)} = \pi_0^* \left(\pi_0^* - \frac{\pi_0^*}{\pi_0^* + \underbrace{\phi}_{U'(C_{T,1})/U'(C_{T,0})} (1 - \pi_0^*)} \right)$$

upward bias in RN vs. physical beliefs induced by risk aversion

room for downward movement when bias collapses at T

Intermediate result

Two-State Setting: Results

Result 1

Under RE, for any DGP,

$$\mathbb{E}[X^*] \leqslant \pi_0^* \left(\pi_0^* - \frac{\pi_0^*}{\pi_0^* + \underbrace{\phi}_{U'(C_{T,1})/U'(C_{T,0})} (1 - \pi_0^*)} \right)$$

Features of bound and interpretation:

1. Relates **observable values** to **unobserved structural parameter**
2. Under risk neutrality ($\phi = 1$): Bound becomes 0
3. Movement in RN beliefs still must correspond (on average) to learning about state, but now have inequality bound where $\frac{\partial \text{bound}}{\partial \phi} > 0$
4. Thus given observed RN belief movement, bound can be inverted to get min. ϕ under RE
5. Bound is conditional on π_0^*, but can take uncond. expectation for implementation

Details

Two-State Setting: Results

Result 1

Under RE, for any DGP,

$$\mathbb{E}[X^*] \leqslant \pi_0^* \left(\pi_0^* - \frac{\pi_0^*}{\pi_0^* + \underbrace{\phi}_{U'(C_{T,1})/U'(C_{T,0})} (1 - \pi_0^*)} \right)$$

Taking $\phi \to \infty$, bound is still well-defined:

Corollary 1

Under RE, for any DGP and any value for ϕ,

$$\mathbb{E}[X^*] \leqslant {\pi_0^*}^2$$

- Can have so much excess vol. that no amount of risk aversion works
- Contrast with Hansen–Jagannathan bound

Two-State Setting: Results

Result 1

Under RE, for any DGP,

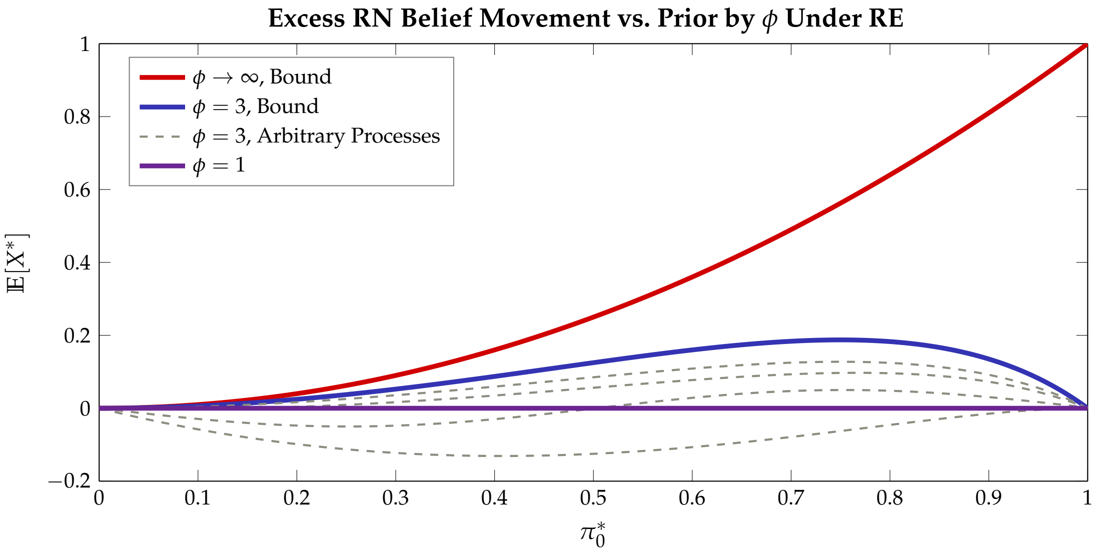
$$\mathbb{E}[X^*] \leqslant \pi_0^* \left(\pi_0^* - \frac{\pi_0^*}{\pi_0^* + \underbrace{\phi}_{U'(C_{T,1})/U'(C_{T,0})} (1 - \pi_0^*)} \right)$$

The bound is tight as $T \to \infty$:

Result 2

There exists a sequence of DGPs, indexed by T, under which $\mathbb{E}[X^*] \to \pi_0^*(\pi_0^* - \pi_0)$ as $T \to \infty$. Meanwhile, for any $T < \infty$, the bound holds with strict inequality as long as $\phi > 1$ and $\pi_0^* \in (0, 1)$.

Graphical Intuition: Bound in Result 1



Bound in General Setting

Result 1 (*General Version*)

Under RE, for any DGP,

$$\widetilde{\mathbb{E}}[X_j^*] \leqslant \widetilde{\pi}_{0,j}^* \left(\widetilde{\pi}_{0,j}^* - \frac{\widetilde{\pi}_{0,j}^*}{\widetilde{\pi}_{0,j}^* + \underbrace{\phi_j}_{\substack{\cancel{U'(C_{T,1})/U'(C_{T,0})} \\ \mathbb{E}_t[M_T \mid R_T^m = \theta_j] / \mathbb{E}_t[M_T \mid R_T^m = \theta_{j+1}]}} (1 - \widetilde{\pi}_{0,j}^*)} \right)$$

General AP framework [*assume discrete prob. space* $(\Omega, \mathcal{F}, \mathbb{P})$, *with filtration* $\{H_t\}$]

- **Setting:** Uncertainty over terminal value of market index, V_T^m
- **State space:** Many return states $\{\theta_j\}$ defined by $R_T^m \equiv V_T^m / V_0^m = \theta_j$
- **Physical beliefs:** $\widetilde{\pi}_{t,j} \equiv \pi_t(R_T^m = \theta_j \mid R_T^m \in \{\theta_j, \theta_{j+1}\})$
- **RN beliefs** (from options): SDF $\{M_t\} \implies \widetilde{\pi}_{t,j}^* = \dfrac{\mathbb{E}_t[M_T \mid R_T^m = \theta_j]}{\mathbb{E}_t[M_T \mid R_T^m \in \{\theta_j, \theta_{j+1}\}]} \widetilde{\pi}_{t,j}$
- **Identification restriction:** ϕ_j is a constant greater than 1

Bound in General Setting

Result 1 (*General Version*)

Under RE, for any DGP,

$$\widetilde{\mathbb{E}}[X_j^*] \leqslant \widetilde{\pi}_{0,j}^* \left(\widetilde{\pi}_{0,j}^* - \frac{\widetilde{\pi}_{0,j}^*}{\widetilde{\pi}_{0,j}^* + \underbrace{\phi_j}_{\substack{\cancel{U'(C_{T,1})/U'(C_{T,0})} \\ \mathbb{E}_t[M_T \mid R_T^m = \theta_j] / \mathbb{E}_t[M_T \mid R_T^m = \theta_{j+1}]}} (1 - \widetilde{\pi}_{0,j}^*)} \right)$$

Corollary 1 ($\mathbb{E}[X^*] \leqslant \pi_0^{*2}$) and Result 2 (bound tightness) also apply in this setting. In addition:

Result 3 (*Interpreting ϕ_j*)

Assume a representative agent with (indirect) utility over the time-T index value, and denote $V_j^m \equiv V_0^m \theta_j$. Then local relative risk aversion $\gamma_j \equiv -V_j^m U''(V_j^m)/U'(V_j^m)$ is given to a first order around return state θ_j by

$$\gamma_j = \frac{\phi_j - 1}{(V_{j+1}^m - V_j^m)/V_j^m} = \frac{\phi_j - 1}{\text{\% return diff. from } \theta_j \text{ to } \theta_{j+1}}$$

General Setting: Assumptions in Detail

Assumption 1 generalizes straightforwardly:

Assumption 1 (*RE — General Case*)

For any $Y \colon \Omega \to \mathbb{R}$, physical beliefs satisfy $\pi_t(Y = y) = \mathbb{P}_t(Y = y)$ with prob. 1 for all t.

But in this setting, need two assumptions on $\phi_{t,j}$ for bound to apply:

Assumption 2 (*Positive Risk Aversion in Index Return*)

$\phi_{t,j} \geqslant 1$ with prob. 1 for all t, j, where return states are ordered such that $\theta_1 < \theta_2 < \cdots < \theta_J$.

In paper:

- What if the agent has an incorrect prior but updates correctly?
- What if $\phi < 1$?

Both cases: Only minor modifications to bounds.

General Setting: Assumptions in Detail

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Assumption 3 (*Constant ϕ_j*)

$\phi_{t,j} = \phi_j$ is constant with prob. 1 for all t and for all *interior* state pairs $\{(\theta_2, \theta_3), \ldots, (\theta_{J-2}, \theta_{J-1})\}$.

General Setting: Assumptions in Detail

Assumption 3 (*Constant ϕ_j*)

$\phi_{t,j} = \phi_j$ is constant with prob. 1 for all t and for all *interior* state pairs $\{(\theta_2, \theta_3), \ldots, (\theta_{J-2}, \theta_{J-1})\}$.

$$\phi_j = \frac{\mathbb{E}_t[M_T \mid R_T^m = \theta_j]}{\mathbb{E}_t[M_T \mid R_T^m = \theta_{j+1}]}$$

Ruled in by Assumption 3:

- Permanent shocks to the SDF [Alvarez & Jermann (2005)]
- Variable rare disasters as in Gabaix (2012): ϕ_j is constant for all but disaster state $j = 1$
- Rep. agent with Epstein–Zin utility, if: (i) $\gamma = 1$; (ii) $\psi = 1$ and Δc_t is an AR(1); or (iii) Δc_t is i.i.d.

Ruled out by Assumption 3:

- Habit formation as in Campbell & Cochrane (1999) **[bug]**
- Heterogeneous beliefs & non-fundamental risk as in Basak (2000) **[feature]**

General Setting: Assumptions in Detail

Assumption 3 (*Constant ϕ_j*)

$\phi_{t,j} = \phi_j$ is constant with prob. 1 for all t and for all *interior* state pairs $\{(\theta_2, \theta_3), \ldots, (\theta_{J-2}, \theta_{J-1})\}$.

$$\phi_{t,j} = \frac{\mathbb{E}_t[M_T \mid R_T^m = \theta_j]}{\mathbb{E}_t[M_T \mid R_T^m = \theta_{j+1}]}$$

If $\phi_{t,j}$ is time-varying, does anything go?

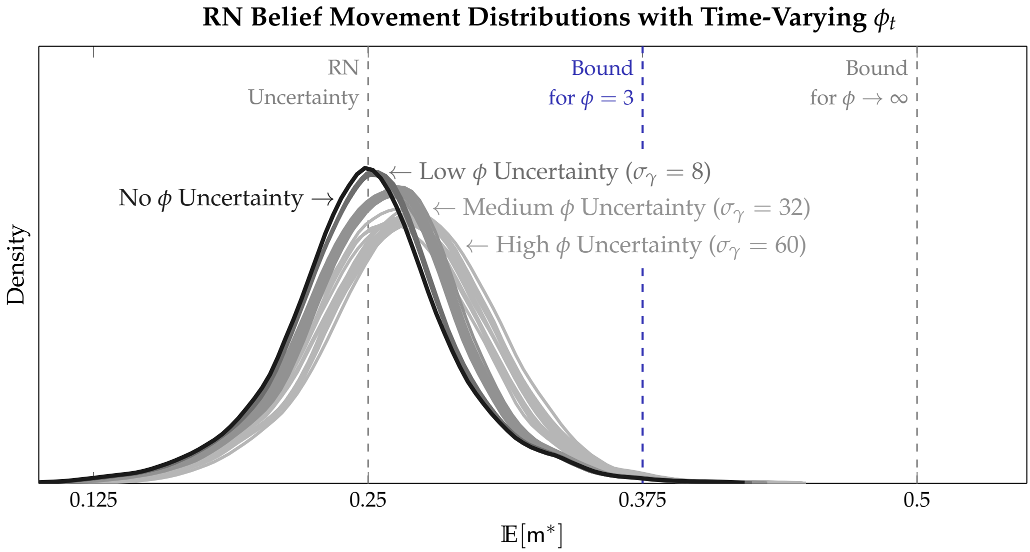
- E.g., assume π_t is constant, but ϕ_t oscillates $1 \to 1.5 \to 1 \to \ldots$
- **This behavior is *also* ruled out by RE: $\phi_{t,j}$ is a ratio of $\mathbb{E}_t[\cdot]$**. Additional result:

Result 4

If ϕ_t evolves as a martingale or supermartingale ($\mathbb{E}_t[\phi_{t+1}] \leqslant \phi_t$), then the previous bounds apply, with ϕ_0 replacing ϕ.

- In paper: Simulations show CC habit model are covered by this result
- Next slide: Simulations extending beyond this supermartingale case

Relaxing the Constant-ϕ Assumption: Simulation Evidence



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   - Data
   - Estimation Results
4. Excess Movement and Aggregate Statistics
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Raw Data and Risk-Neutral Beliefs

Raw data:

- Want distribution over return on market

$\Longrightarrow$ Daily data on S&P 500 index option prices from OptionMetrics, 1996–2018

Details and cleaning [Will use intraday data to address microstructure noise]

Measuring risk-neutral beliefs from options:

- Breeden and Litzenberger (1978): Index price V_T^m has risk-neutral CDF

$$\mathbb{P}_t^*(V_T^m \leqslant v) = 1 + R_{t,T}^f (\underbrace{\partial q_t^m(v)}_{\substack{\text{option price} \\ \text{at strike } v}} / \partial v)$$

- Calculate $\frac{\partial}{\partial v} q_t^m(v)$ numerically following Malz (2014) Details
- **Log excess-return space** (w.r.t. first trading date of option):

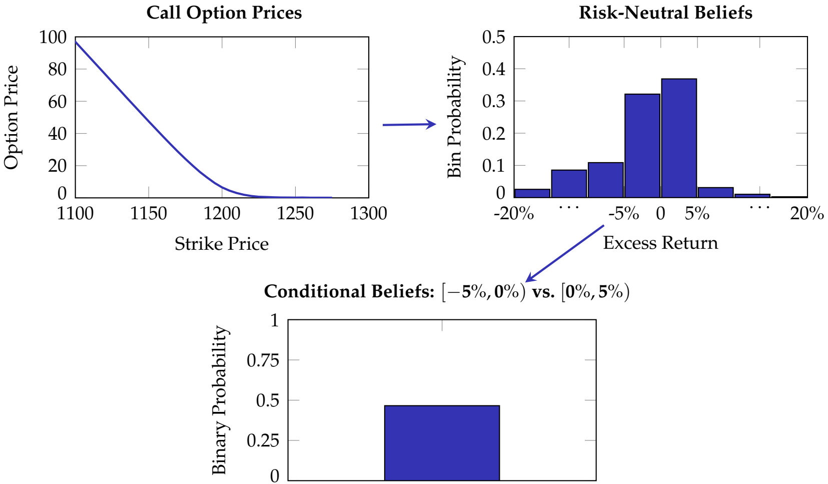
$$\Theta = \{(-\infty, -20\%), [-20\%, -15\%), [-15\%, -10\%), \ldots, [15\%, 20\%), [20\%, \infty)\}$$

  - Again turn all RN beliefs into *conditional* beliefs across adjacent bins
  - Aggregated results: Exclude $(-\infty, -20\%)$, $[20\%, \infty)$ states & consider only interior pairs

Reminder: Empirical Setting

S&P 500 Option Prices and Risk-Neutral Beliefs as of July 1, 2005

Expiration Date: July 16, 2005



Accounting for Market Microstructure Noise

Result 5

Assume that observed $\widehat{\pi}^*_{t,j}$ is measured with error:

$$\widehat{\pi}^*_{t,j} = \widetilde{\pi}^*_{t,j} + \epsilon_{t,j},$$

where $\widetilde{\mathbb{E}}[\epsilon_{t,j}] = 0$, $\widetilde{\mathbb{E}}[\epsilon_{t,j}\,\epsilon_{t+1,j}] = 0$, and $\widetilde{\mathbb{E}}[\epsilon_{t,j}\,\widetilde{\pi}^*_{t,j}] = 0$. Denoting observed one-period expected excess movement by $\widetilde{\mathbb{E}}_t[\widehat{X}^*_{t,t+1,j}]$, we have

$$\widetilde{\mathbb{E}}_t[\widehat{X}^*_{t,t+1,j}] = \widetilde{\mathbb{E}}_t[X^*_{t,t+1,j}] + 2\mathrm{Var}(\epsilon_{t,j}).$$

- Want to "de-noise" excess movement by estimating microstructure error variance $\mathrm{Var}(\epsilon_{t,j})$, then subtracting $2 \times$ estimate

Accounting for Market Microstructure Noise

Result 5

$$\widehat{\pi}^*_{t,j} = \widetilde{\pi}^*_{t,j} + \epsilon_{t,j}$$

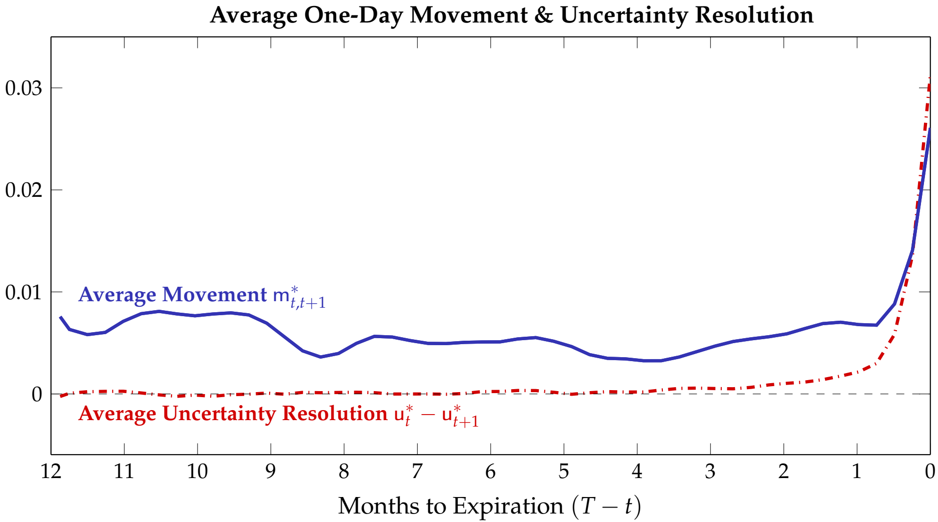
$$\widetilde{\mathbb{E}}_t[\widehat{X}^*_{t,t+1,j}] = \widetilde{\mathbb{E}}_t[X^*_{t,t+1,j}] + 2\mathrm{Var}(\epsilon_{t,j})$$

- How to estimate $\mathrm{Var}(\epsilon_{t,j})$? Use intraday data: Obtain minute-by-minute option price quotes (for random sample of 30 trading days) from CBOE
- First pass: If $\epsilon_{t,j}$ is i.i.d. and true $\widetilde{\pi}^*_{t,j}$ is an Itô process, then $\mathbb{E}[(\widehat{\pi}^*_{t+h,j} - \widehat{\pi}^*_{t,j})^2] \xrightarrow{h \to 0} 2\mathrm{Var}(\epsilon_{t,j})$ [e.g., Zhang, Mykland, Aït-Sahalia (2005)]
- Better version [Li & Linton (*ECMA*, 2021)]: Use non-overlapping windows:

$$\frac{1}{T}\sum_t (\widehat{\pi}^*_{t,j} - \widehat{\pi}^*_{t-k,j})(\widehat{\pi}^*_{t,j} - \widehat{\pi}^*_{t+k,j}) \xrightarrow[\text{(infill)}]{k\to\infty,\, k/T\to 0} \mathrm{Var}(\epsilon_{t,j})$$

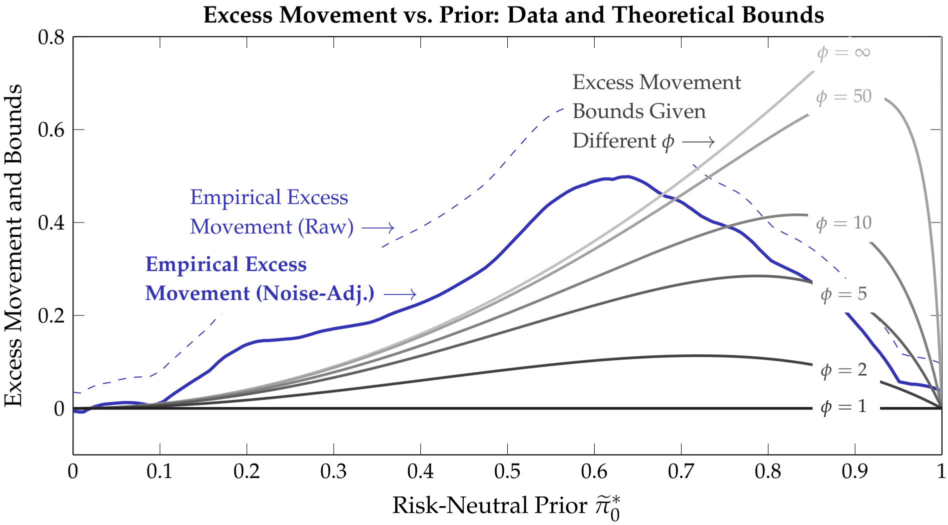
  - Allows for dependent noise and jumps in true $\widetilde{\pi}^*_{t,j}$ (disjoint increments are approx. uncorrelated)
- Estimate separately for each combination of trading day, expiration date, state in our intraday sample, then assign fitted value $\widehat{\mathrm{Var}}(\epsilon_{t,j})$ to end-of-day data to get noise-adjusted excess movement

Summary: Risk-Neutral Belief Variation Within a Contract



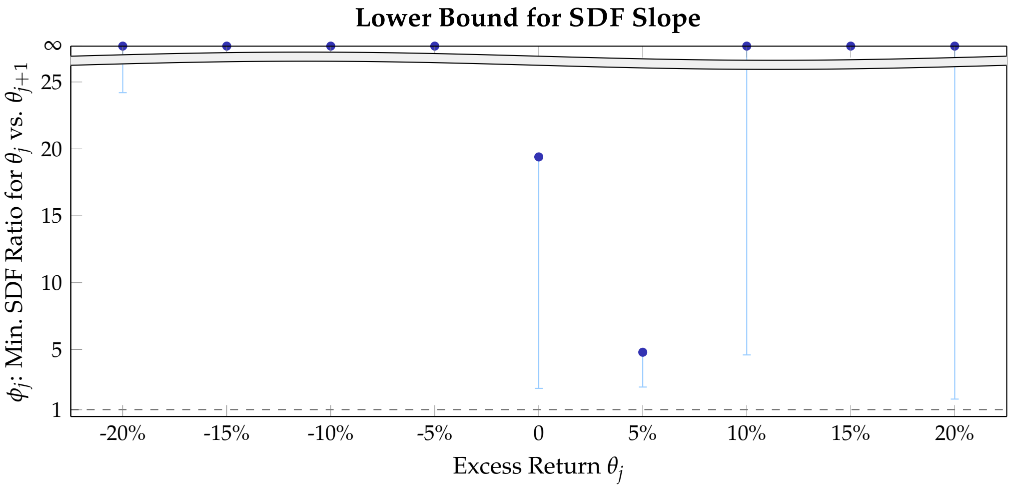
Note: Averages are local means of noise-adjusted data using all expiration dates and interior state pairs.

Summary: Excess Movement over Full Contract



Note: Empirical values are local means and use all expiration dates and interior state pairs.

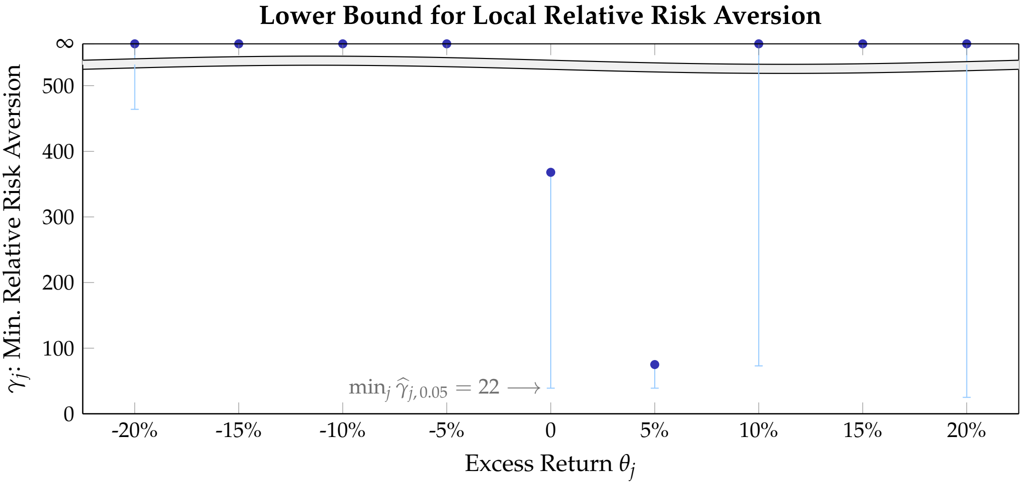
Empirical Implementation of Theoretical Bound



Note: One-sided 95% CIs use block bootstrap and are obtained by inverting a test for ϕ_j.

- **Aggregate across interior states:** $\widehat{\phi} = 54.7$ [CI: $(9.8, \infty)$]

Main Estimation Results: Risk Aversion



- **Aggregate across interior states:** $\widehat{\gamma} = 1{,}075$ [CI: $(175, \infty)$]

Outline

1. Introduction

2. Theory

3. Evidence from Index Options

4. Excess Movement and Aggregate Statistics
   - Financial-Market Predictors
   - SPF Forecasts

5. Discussion and Conclusions

Aggregate Financial-Market Predictors

Regressions for Monthly Average of RN Excess Movement

	(1)	(2)	(3)	(4)	(5)	(6)
Option Bid-Ask Spread	0.24					-0.03
	[0.15]					[0.11]
Option Volume	0.07					-0.05
	[0.09]					[0.10]
RN Belief Stream Length		0.28*			0.16***	0.18**
		[0.14]			[0.05]	[0.07]
VIX^2			0.33*		0.58*	0.62
			[0.16]		[0.32]	[0.36]
Variance Risk Premium			0.38			
			[0.24]			
Vol. of Risk-Aversion Proxy			0.06			
			[0.10]			
Repurchase-Adj. $\lvert pd_t - \overline{pd} \rvert$				0.37***	0.17***	0.18***
				[0.12]	[0.05]	[0.06]
12-Mo. S&P 500 Return				0.30*	0.53**	0.53**
				[0.16]	[0.22]	[0.21]
R^2	0.08	0.08	0.28	0.14	0.37	0.37
Obs.	264	264	264	264	264	264

Notes: *** $p < 0.01$; ** $p < 0.05$; * $p < 0.1$. Heteroskedasticity- and autocorrelation-robust standard errors in brackets, using equal-weighted periodogram estimator with 16 d.o.f. [Lazarus et al. (2018)]. All variables normalized to unit s.d., and all regressions include a constant.

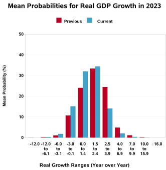
Comovement with SPF Forecast Volatility

Does RN excess movement covary with excess movement in macro forecasts?

- Ultimate goal: survey responses $\longleftrightarrow$ price behavior
- Today: Some promising reduced-form evidence *[feedback welcome]*

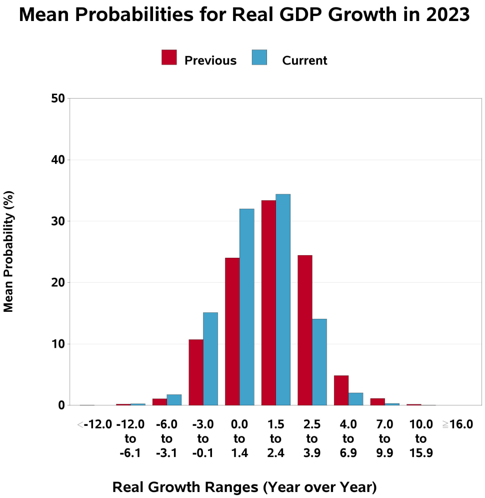
Forecast data:

- Survey of Professional Forecasters (SPF) data from Philly Fed
- **Individual probability forecasts** for future real output growth ["PRGDP"] & GDP deflator ["PRPGDP"]
  - Available 1968Q4 to present *[with some changes in definitions we account for]*
  - Roughly 30–60 participants per survey
  - Survey elicits probabilities for fixed ranges of outcomes for multiple fixed future end dates. . .
  - . . . e.g., for real GDP growth 2022 $\rightarrow$ 2023, mean probabilities as of 2022Q3 (via Philly Fed):



Comovement with SPF Forecast Volatility

- . . . e.g., for real GDP growth 2022 → 2023, mean probabilities as of 2022Q3 (via Philly Fed):



Comovement with SPF Forecast Volatility

Forecast data and excess movement:

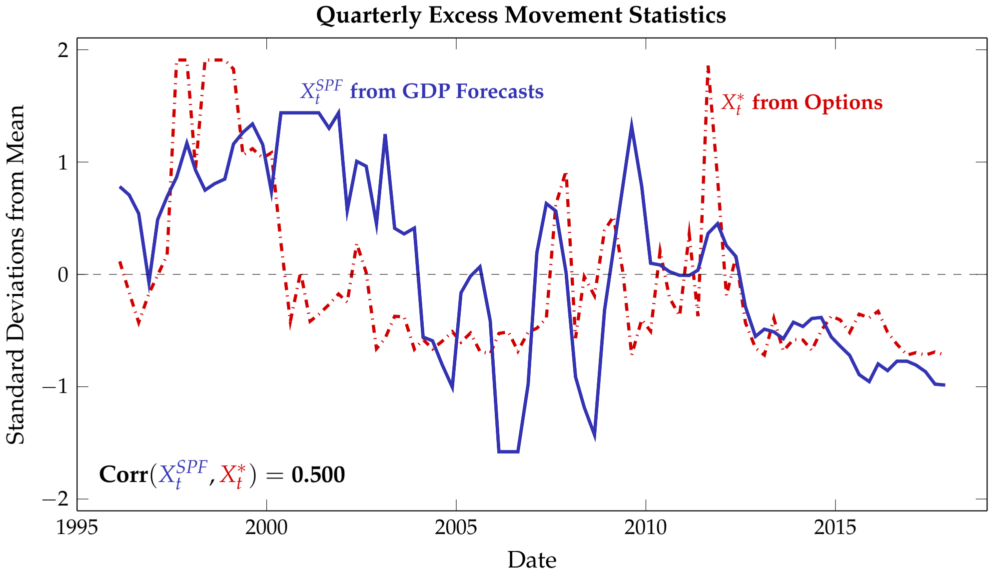
- Survey of Professional Forecasters (SPF) data from Philly Fed
- **Individual probability forecasts** for future real output growth ["PRGDP"] & GDP deflator ["PRPGDP"]
- We keep the forecast end date **fixed** and consider Q-to-Q updates of $\pi_{t,j}$ for each outcome range j
- Then calculate individual-level **excess movement** $X_{t,t+1,j}$ for each quarter and each range

Lemma 2 (*Generalization of Lemma 1*)

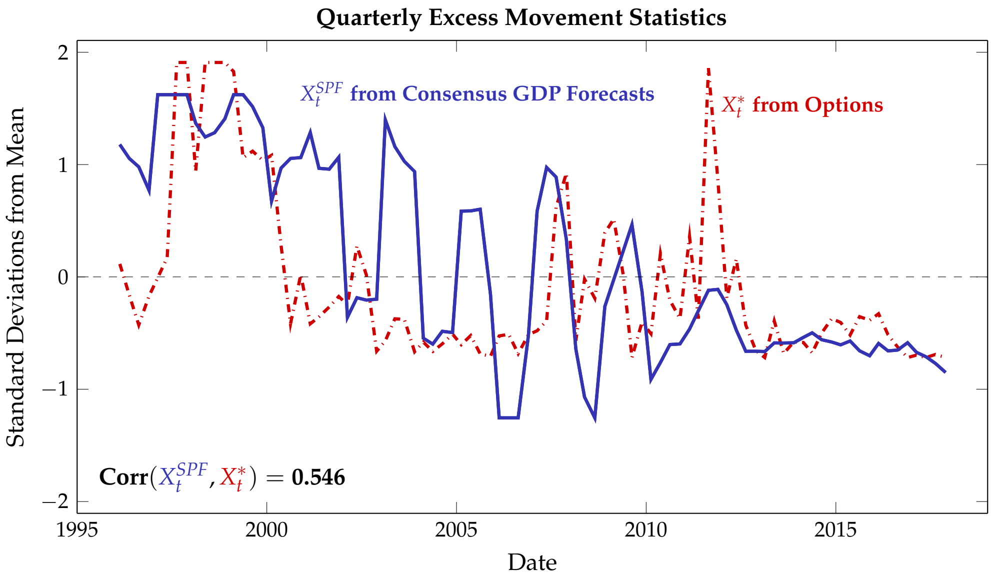
Under RE, for any DGP, $\mathbb{E}[X_{t,t+1,j}] = 0$, where $X_{t,t+1,j} = \mathsf{m}_{t,t+1,j} - (\mathsf{u}_t - \mathsf{u}_{t+1,j})$. This holds for each j, so it holds as well for (i) $X_{t,t+1} \equiv \sum_j X_{t,t+1,j}$, and (ii) the mean of $X_{t,t+1}$ across participants.

- We calculate a 4Q moving average of this mean $X_{t,t+1}$ *[winsorized at 5% on both sides]*
- Denote the resulting statistic X_t^{SPF} for quarter t
- Then compare to that quarter's average of daily RN excess movement in options, X_t^*
- New: Also consider excess movement in **consensus** SPF probabilities for GDP

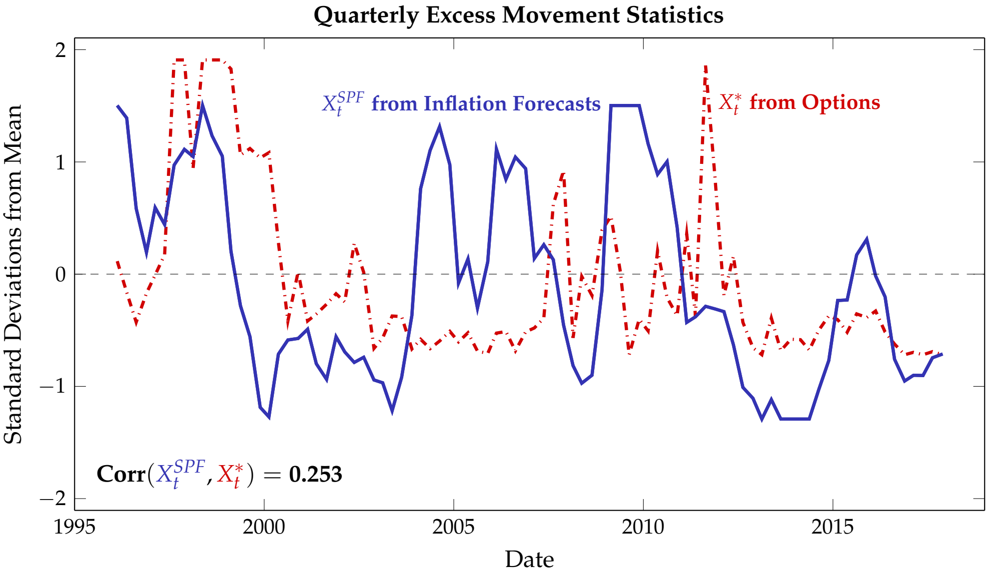
Risk-Neutral and SPF Excess Movement: GDP Growth



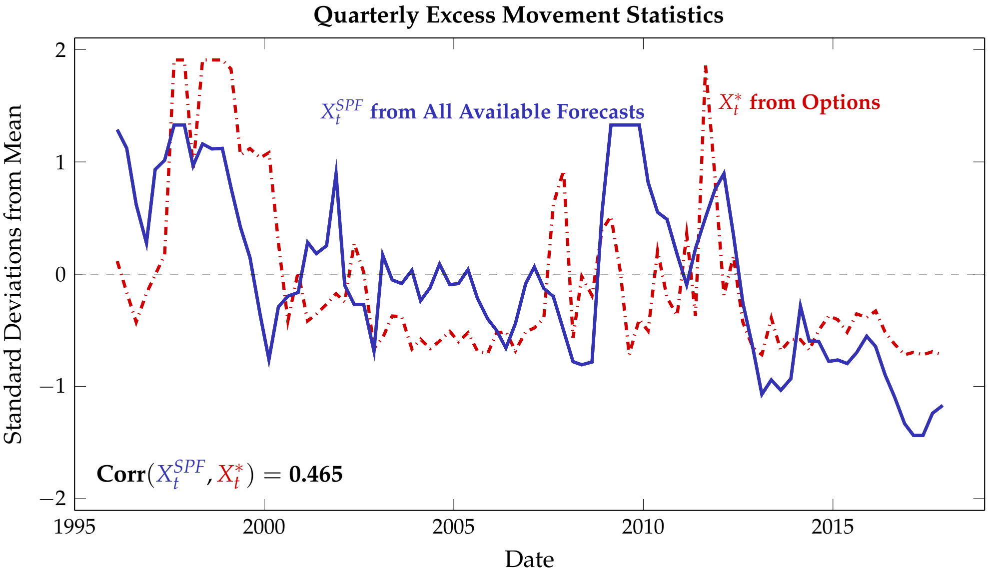
Risk-Neutral and SPF Excess Movement: Consensus GDP Growth



Risk-Neutral and SPF Excess Movement: Inflation



Risk-Neutral and SPF Excess Movement: Average



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Final Notes

Summary:

- New bounds on admissible rational variation in risk-neutral beliefs implied by asset prices
- Bounds do not require keeping track of fundamental value and allow for time-varying discount rates
- Given volatility of observed RN beliefs, bounds are routinely violated in the data
- RE violations appear likely to be responsible at least in part, though can't rule out all possible violations of joint assumptions
- Strong comovement with excess movement in individual SPF forecasts

Some next steps:

- More on empirical correlates & real outcomes
- Long- vs. short-horizon excess movement
- Quantitatively realistic positive models?
- ...

Appendix

Graphical Intuition: Risk-Neutral Beliefs

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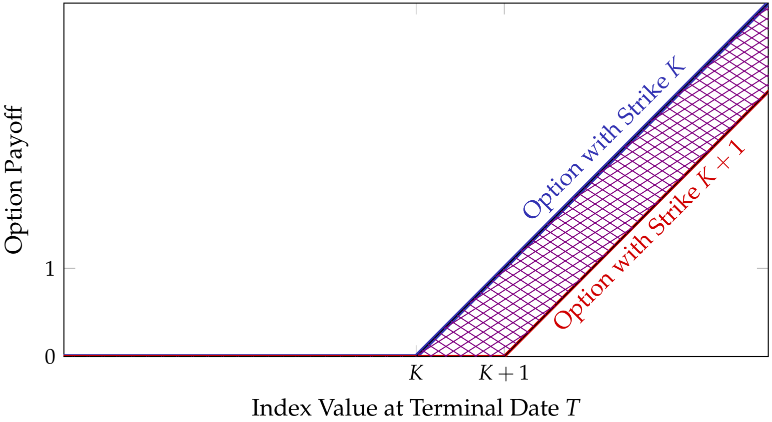
Background: Index option prices $\Longrightarrow$ risk-neutral beliefs over future index price

- Payoff to buying option with strike K + selling strike $K+1 \approx \mathbb{1}\{\text{Index}_T \geqslant K\}$

$$\Longrightarrow \text{Price}_t \approx \mathbb{E}_t^*[\mathbb{1}\{\text{Index}_T \geqslant K\}] = \pi_t^*(\text{Index}_T \geqslant K)$$



Derivation of Lemma 1

Consider the conditional expectation of one-period movement from t_1 to $t_1 + 1$:

$$
\begin{aligned}
\mathbb{E}_{t_1}[\mathsf{m}_{t_1,t_1+1}] &= \mathbb{E}_{t_1}[(\pi_{t_1+1} - \pi_{t_1})^2] \\
&= \mathbb{E}_{t_1}[\pi^2_{t_1+1}] - 2\mathbb{E}_{t_1}[\pi_{t_1+1}]\pi_{t_1} + \pi^2_{t_1} \\
&= \mathbb{E}_{t_1}[\pi^2_{t_1+1}] - 2\pi_{t_1}\pi_{t_1} + \pi^2_{t_1} \quad \text{(martingale property)} \\
&= \mathbb{E}_{t_1}[\pi^2_{t_1+1}] - \pi^2_{t_1} + \pi_{t_1} - \mathbb{E}_{t_1}[\pi_{t_1+1}] \qquad \text{(same)} \\
&= \mathbb{E}_{t_1}[(1 - \pi_{t_1})\pi_{t_1} - (1 - \pi_{t_1+1})\pi_{t_1+1}] = \mathbb{E}_{t_1}[\mathsf{r}_{t_1,t_1+1}],
\end{aligned}
$$

so $\mathbb{E}_{t_1}[X_{t_1,t_1+1}] = 0$. Repeating for all periods and using L.I.E. yields the stated result. $\square$

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Intermediate Result for Bound

Result (*Proposition 1*)

Define $\triangle \equiv \mathbb{E}[X^* \mid \theta = 0] - \mathbb{E}[X^* \mid \theta = 1]$. Under RE, for any DGP,

$$\mathbb{E}[X^*] = (\pi_0^* - \pi_0)\triangle = \left(\pi_0^* - \frac{\pi_0^*}{\phi + (1-\phi)\pi_0^*}\right)(\mathbb{E}[X^*|\theta = 0] - \mathbb{E}[X^*|\theta = 1]) .$$

Key step is in showing $\mathbb{E}^*[X^* \mid \theta] = \mathbb{E}[X^* \mid \theta]$.

Given the above result, the main bound (Proposition 2 in the paper) holds as stated.

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Aggregating Over Belief Streams

Result (*Proposition 8*)

Index belief streams by i, and define $\overline{\phi} \equiv \max_{\pi^*_{0,i}} \mathbb{E}[\phi_i \mid \pi^*_{0,i}]$. Over all streams, under RE,

$$\mathbb{E}[X^*_i] \leqslant \mathbb{E}\left[\left(\pi^*_{0,i} - \frac{\pi^*_{0,i}}{\overline{\phi} + (1-\overline{\phi})\pi^*_{0,i}}\right)\pi^*_{0,i}\right],$$

or, fixing a given $\pi^*_{0,i}$, $\mathbb{E}[X^*_i] \leqslant \left(\pi^*_{0,i} - \frac{\pi^*_{0,i}}{\mathbb{E}[\phi_i]+(1-\mathbb{E}[\phi_i])\pi^*_{0,i}}\right)\pi^*_{0,i}$.

Key point:

- Only observe one draw X^*_i per contract, but $\frac{\partial^2 (\text{bound for } X^*_i)}{\partial \phi_i^2} < 0$, so Jensen's inequality (and L.I.E.) imply the above result
- Therefore min. $\overline{\phi}$ solving the above inequality is lower bound of average ratio of SDF across states $\implies$ info on reasonableness of pricing model required under RE

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General Setting: Details

Previous results can be applied for complete markets.

Now consider general AP case introduced above. Details of setting:

- **Discrete probability space** $(\Omega, \mathcal{F}, \mathbb{P})$, filtration $\{H_t\}$
- **Setting**: Uncertainty over terminal value of market index, V_T^m
- **Return states** $\{\theta_j\}$ defined by $R_T^m \equiv V_T^m / V_0^m = \theta_j$
- **No arbitrage** $\Longrightarrow$ strictly positive SDF $M_{t,T} = M_T / M_t$
- Option prices $\Longrightarrow$ **RN beliefs**: $\pi_t^*(R_T^m = \theta_j) = \frac{\mathbb{E}_t[M_T \mid R_T^m = \theta_j]}{\mathbb{E}_t[M_T]} \pi_t(R_T^m = \theta_j)$
  - Interpret $\pi_t(\cdot)$ as belief of some agent ("the market") observing signals generated by $\mathbb{P}$
  - To map to binary-state setting, localize to **conditional beliefs** for state pair (θ_j, θ_{j+1}):

$$\widetilde{\pi}_{t,j}^* \equiv \pi_t^*(R_T^m = \theta_j \mid R_T^m \in \{\theta_j, \theta_{j+1}\}) = \frac{\phi_j \widetilde{\pi}_{t,j}}{1 + (\phi_j - 1)\widetilde{\pi}_{t,j}},$$

$$\phi_j \equiv \frac{\mathbb{E}_t[M_T \mid R_T^m = \theta_j]}{\mathbb{E}_t[M_T \mid R_T^m = \theta_{j+1}]} \Longrightarrow \text{assume constant } \& \ \phi_j \geqslant 1$$

Raw Data: Details and Cleaning

Details of data:

- End-of-day prices for calls and puts, Jan. 1996–Dec. 2018
- Also obtain underlying index price from OptionMetrics, and hand-collect option settlement values from CBOE
- Calculate risk-free rate using put-call parity following van Binsbergen et al. (2021)

Data cleaning:

- Drop any options with: bids of 0, Black-Scholes implied vol. more than 100%, greater than 6 months to maturity [Constantinides, Jackwerth, Savov (2013)], and any trading date–expiration date combos with fewer than 3 listed prices
- Calculate end-of-day price as average of listed bid and ask prices
- Cleaning for conditional risk-neutral probabilities: to avoid noisy measurement, only use date–state pairs meeting $\pi^*_{t,T_i,j} + \pi^*_{t,T_i,j+1} \geqslant 5\%$

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Spline Details

- Calculate $\frac{\partial}{\partial v} q_{t,T_i}(v)$ numerically following Malz (2014):
  1. Transform call and put price schedules for each date–expiration date set into Black-Scholes IVs
  2. Fit clamped cubic splines to interpolate IVs between strike prices for both calls and puts
  3. Average the calculated call and put IVs at 1,900 strike prices
  4. Invert Black-Scholes implied volatility function to transform resulting IVs back into call prices
  5. Numerically difference the resulting smoothed call-price schedule
- We only use Black-Scholes implied vols for smoothing and then transform vols back into prices, so doesn't require Black-Scholes model to be correct
- "Clamped" cubic spline: Sets slope of IV schedule to be zero at boundary strike-price values, and sets all implied vols below minimum observed strike price to value at minimum price (likewise for max.)
- This guarantees monotonically decreasing and convex call price schedule, which maintains no-arbitrage restrictions
- This is an *interpolating* spline: passes through all observed data (or *knot*) points

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Two-State Example: Non-Constant Discount Rates

What would using the underlying price [Shiller (1981)] give us?

- Consider extreme DGP: No info revealed until date T, so $\pi_0^* = \ldots = \pi_{T-1}^*$
- Price of claim to C_T is

$$\mathbb{E}_t\left[\beta^{T-t}\frac{U'(C_T)}{U'(C_t)}C_T\right]$$

- Consider deterministic consumption stream $C_0 \neq C_1 \neq \ldots$

$\Longrightarrow$ *arbitrary* price variation as C_t changes, but *no* variation in expected payoff C_T

- Paper discusses cases with time-varying risk premia

Robustness: Systematic Mean-Reversion vs. Noise

How real is what we're finding?

- Consider a simple statistical model for risk-neutral beliefs:

$$\widetilde{\pi}^*_{t+1,j} = \mu + \rho(\widetilde{\pi}^*_{t,j} - \mu) + \nu_{t+1}$$

- Setting $\mu = 1/2$, this model yields a prediction that:

$$\mathbb{E}[m^*_{t,t+1,j} - r^*_{t,t+1,j}] = 2(1-\rho)(\widetilde{\pi}^*_{t,j} - 1/2)^2$$

$\Longrightarrow$ should see parabola for excess movement vs. prior

Average Daily Excess Risk-Neutral Belief Movement by Starting Belief